BOUNDS ON LINEAR COMBINATIONS OF INDEPENDENT RANDOM VARIABLES

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ABSTRACT

This paper is concerned with obtaining a tolerance interval (TI) for a linear combination of independent random variables (RV's) in terms of TI's of the individual RV's. Ordinarily, the probability density function (pdf) of the linear combination must be obtained by convolution.

An algorithm is considered which obtains the desired TI without resorting to convolution. The individual TI's are defined relative to a common confidence level. The remainder of the paper is concerned with the condition called "linear conformity" in which the TI formed using the algorithm results in a confidence level of at least that for the individual TI's. Properties of the Fourier transform of the pdf's are related to the linear conformity of the distributions. An important family of transforms are shown to be linearly conformal. Finally, a heuristic procedure is developed such that arbitrary pdf's can be classified with respect to linear conformity.

Key Words: Bounding Algorithm, confidence level, convolution, Fourier transform, probability density function, random variable, tolerance interval.

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1. INTRODUCTION

In many situations of reliability analysis, automated testing and pattern recognition, one is faced with linear combinations of random variables (RV's). For example, in automated testing of nonlinear devices, statistically based interpolation and extrapolation techniques exist [4, 5] which take the form of linear combinations of statistically generated data points. In both reliability analysis and pattern recognition, one may be faced with a decision rule which is a linear function of several independent parameters. Often, the parameters cannot be directly measured, but instead are known through probability distributions. Such a situation exists in process control of integrated circuit fabrication in which process parameters must be held to specific ranges [3].

Consider that a statistical bound on a linear combination of RV's is desired. If the summed RV's are independent, the RV representing the linear combination has a distribution which is obtained by a weighted convolution of the individual probability distribution functions. Each time the linear combination changes, the convolved function changes. When convolution is obtained analytically, a change in one or more distribution parameters or weighting functions corresponds to changing the same parameter in the convolved function expression. Frequently, however, analysis is being done numerically in computer environments, so that the convolution process must be executed when such changes occur. Obviously, applications involving a dynamic analysis in which many parameter changes occur can result in enormous computation time.

In addition to the preceding computational problem, other problems occur in practical situations. The summed RV's often represent physical quantities having some degree of statistical correlation. This dependency, when taken into account, compounds the above computational problem. Additionally, the effort required to determine a probability distribution function can be costly in terms of the amount of data that must be gathered. One may be limited to determining tolerance limits [6] for a physical quantity, since such limits require far less data for their estimation than are required for distribution functions.

This paper examines a "tolerance interval" (TI) bound on a linear combination of RV's determined as a simple combination of TI's of the summed RV's. The process of combination, called the Bounding Algorithm (BA), avoids a direct utilization of probability density functions (pdf's) and thus avoids the above mentioned problems of conventional analytic approaches. The bulk of the paper is concerned with establishing a condition for which the TI formed by the BA is a conservative bound. This condition, called linear conformity, is first established analytically for a family of pdf's. In addition, a heuristic technique is presented which enables one to analyze pdf's not belonging to this family with respect to linear conformity. Statistical dependence between the summed RV's is discussed, and it is argued why the BA results in conservative bounds, even for the dependency case.

2. DEFINITIONS AND ASSUMPTIONS

Consider the linear combination of RV's

$$Y = \sum_{i=1}^{n} C_i X_i \tag{2.1}$$

where the C_i's are constants and the X_i's are RV's. Assume that corresponding to each RV X_i, a continuous pdf $g_i(u)$ exists. Further assume that $g_i(u)$ is symmetric with zero mean. (The requirement of a zero mean is made without loss of generality, since the X_i's and Y may be redefined to accomplish this.) Define the tolerance interval (TI) for X_i with confidence level (CL) α, such that the probability

$$p\{|X_i| \le a_i\} \equiv \int_{-a_i}^{a_i} g_i(u)\,du = \alpha. \tag{2.2}$$

Definition 1 (Tolerance Interval)-The TI for X_i having CL α, written $(-a_i, a_i)$ is defined according to (2.2), where the pdf for X_i is assumed to be symmetric with a zero mean.

Definition 2 (Bounding Algorithm)-Let the TI's for each X_i in (2.1) be defined according to Def. 1, corresponding to a common CL, α. The TI for Y is $(-a_Y, a_Y)$, such that

$$a_Y = \sum_{i=1}^{N} a_i |C_i| \tag{2.3}$$

Definition 3 (Composite Tolerance Interval)-The TI determined according to Def. 2 is called the composite tolerance interval (CTI).

The BA may appear to result in a "worst-case" combination of the individual TI's. A simple example will be given shortly to show that such is not always the case. But first, a condition will be defined regarding the applicability of the BA in obtaining a conservative bound on RV Y of (2.1).

Definition 4 (Linear Conformity)-If a CTI for Y in (2.1) is obtained using the BA, and the CL for each of the individual TI's $[-a_i, a_i]$ is α, then the collection of pdf's for the X_i's is said to be "linearly conformal" (LC) if and only if the CL corresponding to the CTI is at least α for all values of the C_i's, and with no restriction on α.

When the N pdf's are LC, and in addition have the same general distribution form, with perhaps different distribution parameters, this general distribution is said to be "self-linearly conformal" (SLC).

Definition 5 (Conditional Linear Conformity)-If for the previous definition, there exists a minimum value of α for which the CTI CL is at least that of the individual TI's for all values of the C_i's, then the collection of pdf's are said to be "conditionally linearly conformal".

In order to demonstrate that one cannot always assume the condition of LC, consider the discrete probability distribution of Table 1. Suppose RV's X_1 and X_2 have the distribution of Table 1 and are statistically independent. Let $Y=X_1+X_2$. Using the summation

$$p\{Y=j\}=\sum_{i=-10}^{10} p\{X_i=i\}\cdot p\{X_2=j-i\}$$

1. DISTRIBUTION WHICH IS NOT SELF-LINEARLY CONFORMAL

k	p (X=k)
0	.2
± 1	.1
± 2	.05
± 3	.01
± 4	0
± 5	0
± 6	.02
± 7	.05
± 8	.1
± 9	.05
± 10	.02

and noting that $p\{|X_2|>10\}=0$, the distribution for Y is obtained, and is given in Table 2.

2. CONVOLVED DISTRIBUTION CORRESPONDING TO RANDOM VARIABLE Y

k	p (Y=k)	k	p (Y=k)
0	.0968		
± 1	.0750	± 11	.0110
± 2	.0450	± 12	.0034
± 3	.0184	± 13	.0024
± 4	.0083	± 14	.0065
± 5	.0120	± 15	.0120
± 6	.0291	± 16	.0158
± 7	.0494	± 17	.0120
± 8	.0640	± 18	.0065
± 9	.0494	± 19	.0020
± 10	.0290	± 20	.0004

Examination of the two tables reveals that if an α is selected corresponding to $(|X|\leq k, k=0,1,2,3)$ then

$$p\{-2k\leq Y\leq 2k\}<p\{-k\leq (X_1 \wedge X_2)\leq k\}.$$

Therefore the probability distribution of Table 1 is only conditionally LC. This example will be considered later.

The remainder of this paper will be concerned with methods to determine if pdf's are SLC. In order to simplify the investigation of linear conformity, Fourier analysis will be employed in what follows:

3. FOURIER ANALYSIS

Several important properties of pdf's are discussed in this section. These properties pertain to the Fourier transforms of the pdf's, and will be useful in subsequent developments of this paper. One recalls that convolution is the process by which the pdf of a linear combination of independent RV's may be obtained. Since convolution transforms to simple multiplication via the Fourier Integral, it is natural to consider the Fourier transform in investigating linear conformity. The following theorem contains several important properties.

Theorem 1 - Given g(u), the pdf of RV X. If g(u) is symmetrical about the origin of u, then the Fourier transform G(ω) of g(u) has the following properties:

a. $$G(\omega)\Big|_{\omega=0}=1 \qquad (3.1)$$

b. $$\lim_{\omega\to 0_+}\left(dG(\omega)/d\omega\right)+\lim_{\omega\to 0_-}\left(dG(\omega)/d\omega\right)=0 \qquad (3.2)$$

c. $$\sigma^2=-d^2G(\omega)/d\omega^2\Big|_{\omega=0}\geq 0 \qquad (3.3)$$

d. $$G(\omega)\leq 1,\ -\infty<\omega<\infty \qquad (3.4)$$

Proof - The first three parts of the theorem are established from the moment theorem [7],

$$(-j)^k m_k = d^k G(\omega)/d\omega^k \big|_{\omega=0}$$

where $j = (-1)^{1/2}$ and where

$$m_k = \int_{-\infty}^{\infty} u^k g(u)\,du$$

for k = 0, 1, 2, respectively. One need only recognize that for g(u) a pdf with zero mean

$$\int_{-\infty}^{\infty} g(u)\,du=1, \ \int_{-\infty}^{\infty} ug(u)=0, \text{ and } \int_{-\infty}^{\infty} u^2 g(u)\,du=\sigma^2$$

where σ^2 is the variance. The two limits in part a are required for the special case where the derivative of $G(\omega)$ at the origin is discontinuous, as with the Cauchy distribution

$$\beta/\pi(\beta^2+u^2) \leftrightarrow \exp(-\beta|\omega|), \ \beta \geq 0. \qquad (3.5)$$

The last part of the theorem follows from

$$|G(\omega)| = \left|\int_{-\infty}^{\infty} g(u) e^{-j\omega u}\,du\right| \leq \int_{-\infty}^{\infty} g(u)\,du=1.$$ QED

The preceding theorem will be useful when the shape of the function $G(\omega)$ near the origin is considered. One observes that $G(\omega)$ is positive decreasing near the origin, a fact that will be useful shortly.

The next preliminary development concerns an alternative expression of (2.2) utilizing the Fourier transform. (This development is central to the subsequent analysis of this paper.) Assuming as usual that RV X possesses a pdf, symmetric about a zero mean, and assuming that the Fourier transform $G(\omega)$ of g(u) exists, one may replace g(u) by the inverse Fourier transform of $G(\omega)$ in (2.2) [7]:

$$p\{|X|\leq a = \int_{-a}^{a}\left((1/2\pi)\int_{-\infty}^{\infty} G(\omega)e^{j\omega u}d\omega\right)du=\alpha. \quad (3.6)$$

Interchanging the order of integration, and integrating with respect to u, one obtains.

$$p\{|X|\leq a\}=(1/\pi)\int_{-\infty}^{\infty} G(\omega)\frac{\sin\,\omega a}{\omega}\,d\omega. \quad (3.7)$$

It was possible to interchange the order of integration in (3.6) since the integrand was square integrable [7]. That is

$$(1/2\pi)\int_{-\infty}^{\infty}|G^2(\omega)e^{j\omega u}|^2 d\omega \leq (1/2\pi)\int_{-\infty}^{\infty}|G^2(\omega)|d\omega<\infty. \quad (3.8)$$

The right hand inequality can be justified by

$$(1/2\pi)\int_{-\infty}^{\infty}|G^2(\omega)|d\omega=\left(g(u)*g(u)\right)\Big|_{u=0}=h(0)<\infty$$

where g(u) contains no impulses at the origin [7]. h(u) is the convolution of g(u) with itself ("*" denotes convolution), and is also a pdf. Therefore h(0) is finite unless g(u) is a delta function.

Consider the effect of increasing the value of "a" in (3.7). As this variable is increased, the integral must also increase since it directly related to the probability of (3.7). The function $\sin a\omega/\omega$ has the value "a" at the origin. The envelope of this function is $1/\omega$ at some distance from the origin. Increasing "a" also increases the frequency of the sinusoid so that $\sin a\omega/\omega$ approaches the envelope curve more quickly. One observes from the preceding discussion that as "a" is increased, the behavior of $G(\omega)$ near the origin tends to dominate the entire integral.

A similar effect of dominance can be shown for the TI probability corresponding to a pdf convolved with itself. In particular, a linear combination of RV's X_1 and X_2 having a common pdf $g(u)$ has a pdf whose transform is $G^2(\omega)$ [7]. The TI obtained from the BA is $[-2a, 2a]$. The probability expression of (3.7) can be written with obvious substitutions as

$$p\{|X_1+X_2|\leq 2a\}=(1/\pi)\int_{-\infty}^{\infty} G^2(\omega)\ \frac{\sin 2\ \omega a}{\omega}\ d\omega. \tag{3.9}$$

Thus, from the previous discussion of the argument of $\sin \omega a/\omega$, the integral tends to be dominated by a region close to the origin. Also, since in general $G(\omega)$ is a positive monotonically decreasing function near the origin, this decrease toward zero is made more rapid for the square of $G(\omega)$. The integral of (3.9) is dominated by a region closer to the origin than that which dominates the integral of (3.7). The dominant feature just discussed is used in approximating the integrals of (3.7) and (3.9) in Appendix C, where a heuristic classification procedure is developed.

4. EXPONENTIAL TRANSFORM FAMILY

This section considers a family of functions referred to as the "exponential transform family".

Definition 6 (Exponential Transform Family)-A function is a member of the exponential transform family if and only if its Fourier transform can be written as

$$F_{\Theta,k}(\omega)=\exp(-\Theta|\omega^k|), \quad \Theta>0,\ k\geq 0. \tag{4.1}$$

A function belonging to this family is distinguished by the parameters Θ and k.

This family is considered for several reasons. First, the members for k equal to zero, one, and two correspond to the delta, Cauchy, and Normal distributions, respectively. Second, each subset of the family of members with a common value of k, exhibits closure under the operation of convolution in the real domain, and therefore multiplication in the ω domain. This property allows a simplified analysis to be made which is not generally possible for other pdf's. Before proceeding, it should be pointed out that the members of this family do not generally correspond to pdf's. One can show that for integer $k>2$, the function of (4.1) cannot correspond to a non-negative function in the real domain.

The following theorem considers the property of linear conformity with respect to family members which are pdf's.

Theorem 2 - Given the linear combination of N independent RV's, X_i, with zero mean, $Y = \sum_{i=1}^{N} C_i X_i$, where the C_i are constants. If RV's X_i have pdf's which are Fourier transformable as $F_{\Theta,k}$ corresponding to (4.1), with common parameter k, then the collection of these N pdf's is SLC (see Def. 4), if any only if $k \geq 1$. That is, for any α, $0 \leq \alpha \leq 1$, if

$$p\{|X_i| \leq a_i\} = \alpha \tag{4.2}$$

then

$$p\{|Y| \leq \sum_{i=1}^{N} a_i C_i\} \geq \alpha \tag{4.3}$$

if and only if $k \geq 1$.

The proof of Theorem 2 is given in Appendix A.

When dealing with a linear combination of RV's whose distributions correspond to a particular k-member of the exponential transform family, Theorem 2 can be used to determine if the linear sum satisfies the condition of linear conformity. When these RV distributions closely resemble but are not equal to a family member, one may heuristically use the classification associated with this family member.

One may consider alternative algorithms to the BA for combining TI's. The sum-of-squares algorithm (SSA) is often used when the individual RV's of a sum are Normally distributed and independent [1] (consideration of the SSA was suggested by H. D. Helms of the Bell Telephone Laboratories, Holmdel, NJ). In terms of the previous notation, the RV Y is bounded by

$$|Y| \le \left(\sum_{i=1}^{N} a_i^2 C_i^2\right)^{\frac{1}{2}}. \tag{4.4}$$

The following theorem relates the SSA to the property of linear conformity and the exponential transform family.

<u>Theorem 3</u> - Given the linear combination of N independent RV's, X_i, with zero mean, $Y = \sum_{i=1}^{N} C_i X_i$, where the C_i are constants. Let the RV's X_i have pdf's which are Fourier transformable as $F_{\Theta_i,k}$ corresponding to (4.1). Let the TI for each RV be formed as in (2.2), with CL α. Then the TI formed using the SSA will correspond to a CL of at least α, if and only if $k \ge 2$.

The proof of this theorem is outlined in Appendix B. One observes that the SSA yeilds a conservative bound on the linear sum of RV's for the k=2 case only. As pointed out earlier, the exponential transform function does not correspond to a pdf for k>2. Since the k=2 case corresponds to the Normal distribution, it may be argued that the SSA provides a minimum width TI. A formal argument can be given based on an information theoretic approach, but is omitted here for reasons of brevity of the paper. Intuitively, the Normal distribution is a maximum entropy pdf [8].

5. HEURISTIC CLASSIFICATION OF ARBITRARY DISTRIBUTIONS

For the exponential transform family, it was determined that the family member corresponding to $k=1$ was the critical function which partitioned the family into two classes. For $k \geq 1$, the members were SLC, while for $k<1$, they were not SLC. It is natural to look for a classification method which allows one to compare a non-family member to the $k=1$ member in such a way as to determine if this arbitrary function is SLC.

A classification procedure is possible for transforms of arbitrary pdf's which meet certain conditions. This procedure is heuristic due to certain approximations made in its development. These approximations tend to be pessimistic, as seen shortly.

The following notation is used. The standard function $F(\omega)=\exp(-\Theta|\omega|)$ is compared to an arbitrary symmetric function $G(\omega)$. The heuristic procedure considers the linear combination

$$Y = X_1 + X_2 \tag{5.1}$$

in which X_1 and X_2 are RV's having the pdf $g(u)$ which transforms to $G(\omega)$. If the TI for X_1 and X_2 are defined as in (2.2), then the CL corresponding to the TI formed by the BA is given by (3.9). The heuristic procedure considers the region in the positive ω plane for $\omega \leq \pi/a$. There are two classes of functions, $G(\omega)$, for which the procedure gives definite results. Refer to Figure 1-a.

As ω increases, $G(\omega)$ is first less than $F(\omega)$, then intersects $F(\omega)$ at ω_I, and finally is greater than $F(\omega)$. The two functions intersect at $\omega=0$ and ω_I. Note that Θ is chosen so that $\omega_I=\pi/2a$. The function $G(\omega)$ in Figure 1-a will be shown to be not SLC for the particular value of parameter a.

In Figure 1-b the reverse situation is observed. This class of $G(\omega)$ will be shown to be SLC for the particular value of a.

Heuristic Classification Procedure - Sketch $G(\omega)$. Draw an $F(\omega)$ curve such that the two curves intersect at a particular ω_I. At each ω_I such that $G(\omega_I)$ is between 1 and .25, $G(\omega)$ is SLC if the situation of 1-b is observed, or is not SLC if the situation of Figure 1-a is observed. Each of the ω_I corresponds to a particular value of the tolerance limit $a=\pi/2\omega_I$. Usually, the function $G(\omega)$ will either belong entirely to one of the two classes. Also, if more than one intersection occurs between $F(\omega)$ and $G(\omega)$ for a specific value of "a", the heuristic is indeterminate at that point.

Once the point at which $G(\omega)$ is equal to .25 is reached (the smallest $|\omega|$ such that $G(\omega) = .25$), the heuristic is slightly modified. Providing that only one intersection occurs in the interval $0<\omega<\pi/a$, $G(\omega)$ is classified as SLC or not SLC on the basis of the classification at $\omega_{.25}$, where $G(\omega_{.25})=.25$, for a $\leq\pi/2\omega_{.25}$. Using the heuristic over the positive ω domain in this manner, one determines the ranges of value "a" for which $G(\omega)$ is SLC.

Although the heuristic requires that many $F(\omega)$ curves be constructed, the $G(\omega)$ curve can usually be analyzed by inspection using at most a few $F(\omega)$ curves. One approach to analysis would be to have a standard chart drawn with a family of $\exp(-\Theta|\omega|)$ curves. One would then draw the $G(\omega)$ curve normalized with respect to a convenient scaling of ω. This scaling would allow $G(\omega)$ to be fit on the standard chart.

The development of the heuristic is given in Appendix C. As this derivation assumes equal weights of the RV's in (5.1), one may find the conditions of this analysis severely limiting. However, for many distributions, it is possible to show that the equal weighted linear combination represents either the minimum or the maximum probability combination with respect to the relative variation of weights.

Consider the linear combination

$$Y = X_1 + C\, X_2. \tag{5.2}$$

This combination represents the various relative weightings of the two RV's X_1 and X_2, where the weight of X_1 is unity without loss of generality. For C=0, the probability associated with the TI of Y is obviously that of X_1. Assuming that X_1 and X_2 have identical pdf's, the BA is applied as usual for some CL on X_1 and X_2. The actual CL of the TI formed by the algorithm is, from (2.3) and (3.7).

$$p\{|X_1+CX_2| \leq a+C\cdot a\} = (2/\pi)\int_0^\infty G(\omega)\,G(C\omega)\,\frac{\sin(a(1+C)\omega)}{\omega}\,d\omega \tag{5.3}$$

where $G(\omega)$ is the pdf transform for X_1 and X_2, and "a" is the TI constant of (2.2). One must show that the probability of (5.3) is a min-max value for C=1. Substitute $z=(1+C)\omega$ in (5.3) and differentiate with respect to C. Denoting the derivative by a prime, one obtains the right-hand side of (5.3) as

$$(2/\pi)\int_0^\infty G'(z/1+C)\,G(Cz/1+C)\,(-z/(1+C)^2)\ \frac{\sin az\,dz}{z}$$

$$+\ (2/\pi)\int_0^\infty G(z/1+C)\,G'(Cz/1+C)\,(z/(1+C)^2)\ \frac{\sin az\,dz}{z},$$

which is zero for C=1. Although it is not possible to show that this is the only zero for a general $G(\omega)$ transform, it is the only zero when $G(\omega)$ is an exponential transform family function. One may heuristically use the C=1 min-max point for the worst (or best) case in classifying $G(\omega)$.

As a final consideration of this section, the heuristic procedure is generalized.

Theorem 4 - Let $G_i(\omega)$ be any N non-negative transforms of pdf's $g_i(u)$ respectively. If $g_i(u)$ are each classified by the heuristic as SLC, then the N-fold convolution of the $g_i(u)$ will be classified by the heuristic procedure as SLC.

Conversely, if $g_i(u)$ are each classified as not SLC, then the N-fold convolution of $g_i(u)$ will be classified as not SLC. Theorem 4 indicates that for a summation of N RV's each having SLC pdf's, the pdf of the sum will also be SLC. For pdf's satisfying the Central Limit Theorem [6], this result is expected since the pdf of the sum approaches a Normal pdf as N increases. However, Theorem 4 goes further by handling pdf's with unbounded variances (Theorem 4 is proved in Appendix D).

6. CONCLUSIONS AND AN EXAMPLE

The BA, defined in this paper, provides a simple means of obtaining a TI for a linear sum of independent RV's in terms of the TI's of the summed RV's. The condition of linear conformity was defined as the situation wherein the CTI of the BA was a conservative TI, for a given CL.

The investigation of conditions leading to a conservative bound on the linear sum has been limited to the case where the pdf's of the summed RV's are of the same general form. The heuristic procedure for analyzing arbitrary pdf's was limited to determining the condition of SLC pdf's. It is the conjecture of the authors that if each of several pdf's are SLC, then the linear combination of RV's having these pdf's is LC. This conjecture has not yielded to direct verification by the authors, although experience with many pdf's seems to indicate the validty of this hypothesis.

When some of the RV's combined in a linear sum are monotonically dependent, the CTI can be shown to be conservative [2]. When the RV's are linearly dependent, then the CTI is no longer conservative, but rather an exact computation of the TI for the linear sum. One concludes that the assumption of statistical independence made in this paper results in a conservative analysis. Statistical dependence does not, therefore, invalidate the results obtained.

The exponential transform family was defined. This family of functions was shown to be partitioned into two classes with respect to the property of linear conformity. By comparing other functions to members of the family, one may deduce whether the compared functions exhibit the property of linear conformity. The exponential transform family contains the delta, Cauchy, and Normal pdf's, and is therefore important in its own right. Finally, a heuristic method was developed which enables the classification of a much larger number of functions by a simple graphical analysis.

For some $G(\omega)$ functions, the heuristic is indeterminate under conditions previously discussed. Even when such an indeterminacy exists, there is some maximum ω_I (and therefore some minimum "a") for which the heuristic is determinate. In order to illustrate the application of the heuristic for the indeterminate case, the following example is given.

Consider the transform pair

$$.234\exp(-9|u|)+.12\left(\exp(-|u-.81|)+\exp(-|u+.81|)\right)$$
$$\leftrightarrow .4212/(\omega^2+.81) + .48 \cos (8\omega)/(\omega^2+1) \qquad (6.1)$$

The pdf given in (6.1) is a continuous approximation to the discrete example of Table 1. Figure 2 illustrates the heuristic method applied to (6.1). Note that a semi-log plot was used, so that the standard function of the heuristic is a straight line drawn from the point (1,0) through the $G(\omega)$ curve. The intersection of this standard curve results in determining for $\pi/2a\leq.24$. That is, for an intersection at $\pi/2a$, no other intersection occurs for $\omega<\pi/a$. The minimum value of the tolerance limit "a" is therefore $a_{min}=\pi/.48=6.545$. $G(\omega)$, given in (6.1), is SLC for $a>6.545$.

To compare the heuristic result with an analytic analysis, one must determine when

$$(1/\pi)\int_{-\infty}^{\infty} G(\omega)\frac{\sin \omega a}{\omega}\, d\omega \leq (1/\pi)\int_{-\infty}^{\infty} G^2(\omega)\frac{\sin 2\omega a}{\omega}\, d\omega \qquad (6.2)$$

is true. Using complex integration and numerical analysis, one may show that the inequality of (6.2) is satisfied for $a \geq 3.6047$ (the analysis is lengthy and is therefore omitted here). To determine how conservative the heuristic result is, one may compute the probabilities which correspond to a_{min} and $a_{analytic} = 3.6047$. Using (2.2) with g(u) given by (6.1) yields

$$p\{|X| \leq a_{min}\} \simeq .575, \text{ and } p\{|X| \leq a_{analytic} \simeq .503.$$

The heuristic method has resulted in a slightly conservative estimate of the actual CL for which the BA provides a conservative TI.

As a final note of this paper, it is not to be assumed that $G(\omega)$ need be non-negative for the heuristic to work. Indeed the transform of the rectangular distribution is of the form $p_T(u) \leftrightarrow \sin\omega T/\omega T$, where the pdf is centered about the u origin, and has width 2T. This pdf can be shown to be SLC. The heuristic method determines that $p_T(u)$ is SLC, with no indeterminate region.

APPENDIX A. PROOF OF THEOREM 2

Consider an inductive proof. For the N=2 case, it is assumed that, from (4.2)

$$(1/\pi)\int_{-\infty}^{\infty} \exp(-\Theta_1|\omega^k|)\sin\omega a_1\, d\omega/\omega = (1/\pi)\int_{-\infty}^{\infty} \exp(-\Theta_2|\omega^k|)\sin\omega a_2\, d\omega/\omega. \qquad (A.1)$$

Substituting $a_1 Z=a_2 \omega$ and multiplying through by π, one obtains

$$\int_{-\infty}^{\infty} \exp(-\Theta_1 |\omega^k| \sin\omega a_1 \, d\omega/\omega$$

$$= \int_{-\infty}^{\infty} \exp(-\Theta_2 (a_1/a_2)^k |Z^k| \sin Z a_1 \, dZ/Z \qquad \text{(A.2)}$$

(A.2) implies that $\Theta_2 = \Theta_1/(a_1/a_2)^k$. Consider the pdf transform for RV $Y = C_1X_1+C_2X_2$. From the Convolution Theorem [7], one may write the transform of $g_Y(u)$, the pdf of Y, as

$$g_Y(u) \leftrightarrow G_1(C_1\omega)\cdot G_2(C_2\omega) = \exp(-\Theta_1 |(C_1\omega)^k| - \Theta_2 |(C_2\omega)^k|)$$

$$= \exp(-(\Theta_1 C_1^k + \Theta_2 C_2^k)|\omega^k|). \qquad \text{(A.3)}$$

Using the above expression for Θ_2, (A.3) becomes

$$g_Y(u) \leftrightarrow \exp(-\Theta_1 (C_1^k + C_2^k (a_2/a_1)^k)|\omega^k|) = \exp(-\Theta_1 \gamma^k |\omega^k|) \qquad \text{(A.4)}$$

where $\gamma^k = C_1^k + (C_2 a_2/a_1)^k$. Using (3.7), the probability of the CTI is

$$p\{|Y| \le a_1C_1 + a_2C_2\} = (1/\pi)\int_{-\infty}^{\infty} \exp(-\Theta_1 \gamma^k |\omega^k|) \sin((C_1a_1 + C_2a_2)\omega) \, d\omega/\omega.$$

Substituting $v=\gamma\omega$, the last equation becomes

$$p\{|Y| \le a_1C_1 + a_2C_2\} = (1/\pi)\int_{-\infty}^{\infty} \exp(-\Theta_1 |v^k|) \sin((C_1a_1 + C_2a_2)v/\gamma) \, dv/v \qquad \text{(A.5)}$$

From (3.7) and the properties of a pdf, the integral of (A.5) increases monotonically with the constant $(C_1a_1+C_2a_2)/\gamma$. The right-hand side of (A.5) has the same form of the integrals in (A.1). Therefore, the proof is concerned with determining the conditions such that

$$(C_1 a_1 + C_2 a_2)/\gamma \geq a_1 . \tag{A.6}$$

Substituting the definition of γ and $a_2 = (\Theta_2/\Theta_1)^{1/k} a_1$ from the expression of Θ_2, (A.6) becomes after some algebraic manipulation

$$a_1 \left(1+(C_2/C_1)(\Theta_2/\Theta_1)^{1/k}\right) / \left(1+(C_2/C_1)^k (\Theta_2/\Theta_1)\right)^{1/k} \geq a_1 . \tag{A.7}$$

The constants in (A.7) are all non-negative, so the inequality can be written, after raising both sides to the k^{th} power as,

$$\left(1+(C_2/C_1)^k (\Theta_2/\Theta_1)^{1/k}\right)^k \geq 1+(C_2/C_1)^k (\Theta_2/\Theta_1) . \tag{A.8}$$

Let $\zeta^k = (C_2/C_1)^k (\Theta_2/\Theta_1)$. Thus (A.8) is $(1+\zeta)^k \geq 1+\zeta^k$. Since ζ is a non-negative constant, it is clear that the inequality is satisfied when $k \geq 1$. This proves the N=2 case for the proof.

The inductive step assumes the N-case holds. That is, it is assumed that

$$p\{|\textstyle\sum_{i=1}^{N} C_i X_i| \leq \sum_{i=1}^{N} C_i a_i\} \geq \alpha . \tag{A.9}$$

The N+1 case is concerned with the linear combination

$$Y = \sum_{i=1}^{N+1} C_i X_i = \sum_{i=1}^{N} C_i X_i + C_{N+1} X_{N+1} = \hat{X} + C_{N+1} X_{N+1} . \tag{A.10}$$

The transform of the pdf for $\hat{X}$ is again that of the individual X_i pdf form, and can be written as $\exp(-\hat{\Theta}|\omega^k|)$. Define the constants $\hat{a} = \sum_{i=1}^{N} C_i a_i$. By the assumption that the N-case holds,

$$p\{|\hat{X}| \leq \hat{a}\} \geq \alpha, \text{ if } k \geq 1$$
$$< \alpha, \text{ if } k < 1 .$$

It is clear that there exists another constant, $\tilde{a}$, such that $p\{|\hat{X}|\leq\tilde{a}\}=\alpha$. Then

$$\tilde{a}\leq\hat{a}, \text{ if } k\geq 1$$

$$\tilde{a}>\hat{a}, \text{ if } k<1.$$

(A.10) corresponds to the N=2 case. From the first part of the proof, it follows that

$$p\{|Y|\leq\tilde{a}+a_{N+1}C_{N+1} \begin{array}{l}\geq\alpha, \ k\geq 1\\ <\alpha, \ k<1.\end{array}$$

But

$$\tilde{a}+a_{N+1}C_{N+1}\underset{>}{\leq}\hat{a}+a_{N+1}C_{N+1}=\sum_{i=1}^{N+1}C_i a_i, \begin{array}{l} k\geq 1\\ k<1.\end{array}$$

Therefore, it follows that

$$p\{|Y|\leq\hat{a}+a_{N+1}C_{N+1}\}\underset{<}{\geq}p\{|Y|\leq\tilde{a}+a_{N+1}C_{N+1}\}\begin{array}{l}\geq\alpha, \ k\geq 1\\ <\alpha, \ k<1.\end{array}$$

Therefore, the N+1 case holds. That is, the N+1 collection of pdf's of the N+1 RV's is SLC if and only if $k\geq 1$. QED

APPENDIX B. OUTLINE OF THEOREM 3 PROOF

The proof of Theorem 3 parallels the proof of Appendix A. One uses the expression $(a_1^2C_1^2+a_2^2C_2^2)^{\frac{1}{2}}$ in place of $(a_1C_1+a_2C_2)$. The condition to be determined now is when $(C_1^2a_1^2+C_2^2a_2^2)^{\frac{1}{2}}/\gamma\geq a_1$ is true. By similar algebraic manipulations and the definition $\zeta^k=(C_2/C_1)^k(\theta_2/\theta_1)$, the above inequality becomes $a_1\left(1+\zeta^2\right)^{\frac{1}{2}}/\left(1+\zeta^k\right)^{1/k}\geq a_1$. Analysis of this inequality will reveal that the sense of the inequality holds if and only if $k\geq 2$. The inductive step also parallels that of Appendix A.

APPENDIX C. DEVELOPMENT OF THE HEURISTIC CLASSIFICATION METHOD

The heuristic method is concerned with determining when

$$(1/\pi)\int_{-\infty}^{\infty} G^2(\omega)\sin(2\omega a)\,d\omega/\omega \geq (1/\pi)\int_{-\infty}^{\infty} G(\omega)\sin(\omega a)\,d\omega/\omega \qquad \text{(C.1)}$$

is true for the function $G(\omega)$. Several approximations are made in the following analysis. The first approximation is that only the first half-period of the sinc function (that is, the function $\sin\omega a/\omega$) is considered. This approximation is based on the fact that the sinc function falls off as $1/\omega$ and that the magnitude of $G(\omega)$ decreases monotonically for many pdf transforms.

A second approximation is that $\sin 2\omega a/\omega$ and $\sin\omega a/\omega$ are each replaced by a rectangular function having the same area under them as the original functions and with width equal to the corresponding half-period. It is noted that this second approximation is exact for $G(\omega)=\exp(-\Theta|\omega|$

Incorporating the two approximations and dividing out constants, one obtains the approximated representation of (C.1) for symmetric $G(\omega)$, as

$$\int_{0}^{\pi/2a} G^2(\omega)\,d\omega \geq (1/2)\int_{0}^{\pi/a} G(\omega)\,d\omega \qquad \text{(C.2)}$$

The analysis considers the two classes of $G(\omega)$ represented in Figure 1. It is to be shown that Figure 1-a corresponds to the invalidity of (C.2), while Figure 1-b corresponds to the inequality being true. For each of these two classes, two cases are considered. The first case is when $G(\pi/2a)\geq .25$, and the second is when $G(\pi/2a)\leq .25$.

Consider the $G(\pi/2a)\geq.25$ case for both classes, shown in Figure 1. At some ω_I such that $G(\omega_I)\geq.25$, construct a standard function, $F(\omega)=\exp(-\Theta|\omega|)$ such that it intersects $G(\omega)$ at ω_I. Define

$$\Delta(\omega)=G(\omega)-F(\omega). \tag{C.3}$$

The intersection point is to correspond to $\pi/2a$. Substituting (C.3) in (C.2), one obtains

$$\int_0^{\pi/2a}\left(F(\omega)+\Delta(\omega)\right)^2 d\omega \overset{?}{\geq} (1/2)\int_0^{\pi/a}\left(F(\omega)+\Delta(\omega)\right)d\omega \tag{C.4}$$

where the direction of the inequality is yet unknown. One easily shows that

$$\int_0^{\pi/2a} F^2(\omega)\,d\omega = (1/2)\int_0^{\pi/a} F(\omega)\,d\omega. \tag{C.5}$$

Expanding (C.4) and subtracting (C.5), one obtains

$$\int_0^{\pi/2a}\Delta(\omega)\left\{\Delta(\omega)+2F(\omega)\right\}d\omega \overset{?}{\geq} (1/2)\int_0^{\pi/2a}\Delta(\omega)\,d\omega+(1/2)\int_{\pi/2a}^{\pi/a}\Delta(\omega)\,d\omega, \text{ or}$$

$$\int_0^{\pi/2a}\Delta(\omega)\left(\Delta(\omega)+2F(\omega)-\tfrac{1}{2}\right)d\omega-(1/2)\int_{\pi/2a}^{\pi/a}\Delta(\omega)\,d\omega \overset{?}{\geq} 0. \tag{C.6}$$

Since $G(\omega)$ and $F(\omega)$ are at least .25 in $0\leq\omega\leq\pi/2a$,

$G(\omega)+F(\omega)-1/2\geq 0$, $0<\omega<\pi/2a$.

In Figure 1-a, $\Delta(\omega)\leq 0$ for $0\leq\omega\leq\pi/2a$, and $\Delta(\omega)\geq 0$ for $\pi/2a\leq\omega\leq\pi/a$. In this situation, both integrals of (C.6) are non-positive. Therefore the inequality is not satisfied. $G(\omega)$ is not SLC.

Now consider Figure 1-b and (C.6). Here $\Delta(\omega)\geq 0$ for $0<\omega<\pi/a$, and $\Delta(\omega)\leq 0$ for $\pi/2a\leq\omega\leq\pi/a$. Both integrals of (C.6) are non-negative, and $G(\omega)$ is SLC.

The case where $G(\pi/2a) \leq .25$ is now considered. $F(\omega)$ is now constructed so that it intersects $G(\omega)$ at $G(\omega_I) = .25$ (if $G(\omega) = .25$ at more than one point, $\omega_{\frac{1}{4}}$ is the smallest value of corresponding to $G(\omega) = .25$). (C.4) is now written, using (C.3), as

$$\int_0^{\omega_{\frac{1}{4}}} (F(\omega)+\Delta(\omega))^2 d\omega + \int_{\omega_{\frac{1}{4}}}^{\pi/2a} (F(\omega)+\Delta(\omega))^2 d\omega \overset{?}{\geq} (1/2)\int_0^{\pi/a} (F(\omega)+\Delta(\omega)) d\omega. \tag{C.7}$$

Subtracting (C.5), bringing all terms on the left-hand side, and using (C.3), one obtains

$$\int_0^{\omega_{\frac{1}{4}}} \Delta(\omega)\left(F(\omega)+G(\omega)-(1/2)\right) d\omega + \int_{\omega_{\frac{1}{4}}}^{\pi/a} \Delta(\omega)\left(F(\omega)+G(\omega)-(1/2)\right) d\omega$$

$$-(1/2)\int_{\pi/2a}^{\pi/a} (\Delta(\omega)) d\omega \overset{?}{\geq} 0. \tag{C.8}$$

Since $F(\omega)$ and $G(\omega)$ are at least .25 for $\omega \leq \omega_{\frac{1}{4}}$, and less than .25 for $\omega_{\frac{1}{4}} < \omega \leq \pi/2a$,

$$F(\omega)+G(\omega)-(1/2) \geq 0, \quad \omega \leq \omega_{\frac{1}{4}}$$

$$\leq 0, \quad \omega_{\frac{1}{4}} \leq \omega \leq \pi/2a.$$

In Figure 1-a, $\Delta(\omega) \leq 0$ for $\omega \leq \omega_{\frac{1}{4}}$ and $\Delta(\omega) \geq 0$ for $\omega \geq \omega_{\frac{1}{4}}$. The three integrals of (C.8) are non-positive. Therefore, the inequality is not satisfied. $G(\omega)$ is not SLC.

Now consider Figure 1-b and (C.8). Here $\Delta(\omega) \geq 0$ for $\omega \leq \omega_{\frac{1}{4}}$ and $\Delta(\omega) \leq 0$ for $\omega \geq \omega_{\frac{1}{4}}$. The three integrals are non-negative, satisfying the inequality. $G(\omega)$ is therefore SLC.

As a final note, one realizes that the above analysis is conservative. This is true since the integrands of (C.6) and (C.8) were required to be either non-negative or non-positive throughout the range of integration. This requirement is clearly a worst-case situation.

APPENDIX D. PROOF OF THEOREM 4

Let the pdf's $g_i(u)$ be each classified SLC by the heuristic, and let their transforms be denoted $G_i(\omega)$. Then at any ω, say ω_I, there exists standard functions $F_i(\omega)$ such that

$$F_i(\omega_I)=\exp(-\Theta_i|\omega_I|)=G_i(\omega_I).$$

By assuming $g_i(u)$ are classified as SLC, it follows that $G_i(\omega)\geq\exp(-\Theta_i|\omega|)$ for $\omega\leq\omega_I$, and $G_i(\omega)<\exp(-\Theta_i|\omega|)$ for $\omega\geq\omega_I$. But

$$\prod_{i=1}^{N} G_i(\omega)=\prod_{i=1}^{N}\exp(-\Theta_i|\omega_I|)=\exp(-\sum_{i=1}^{N}\Theta_i|\omega_I|) \quad (D.1)$$

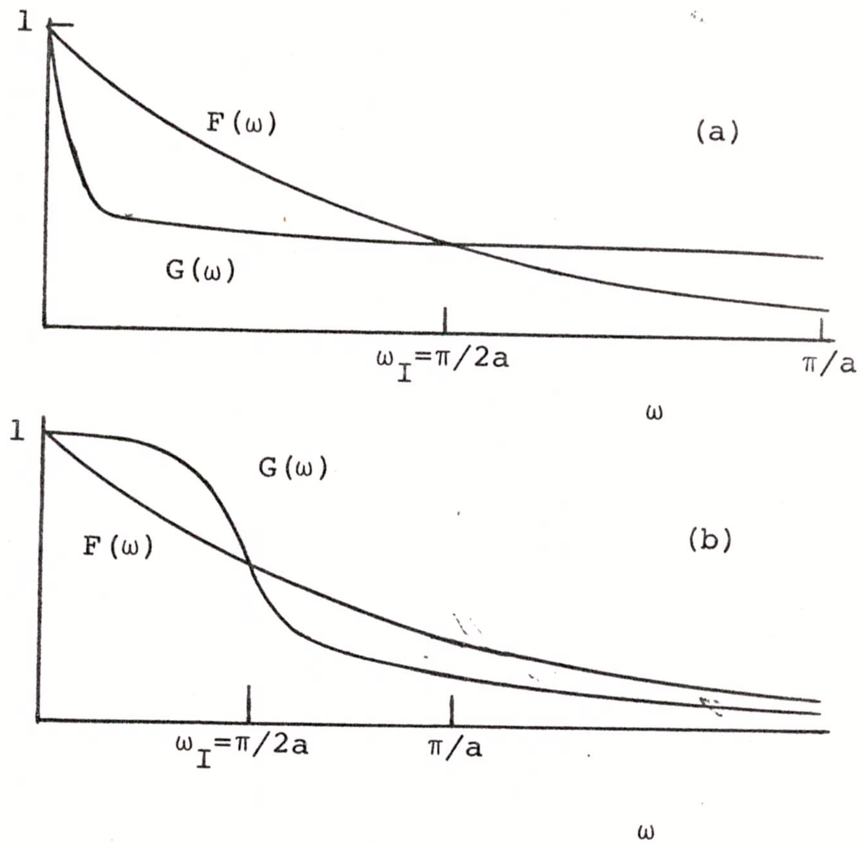
and

$$\prod_{i=1}^{N} G_i(\omega)\geq\exp(-\sum_{i=1}^{N}\Theta_i|\omega|),\quad \omega<\underline{\omega}_I$$
$$\leq \qquad , \quad \omega\geq\omega_I. \quad (D.2)$$

One recognizes (D.1) and (D.2) as the criteria used by the heuristic to classify a pdf with transform $\prod_{i=1}^{N} G_i(\omega)$, where the standard function has parameter $\Theta_T=\sum_{i=1}^{N}\Theta_i$. Since $\prod_{i=1}^{N} G_i(\omega)$ is the transform of the N-fold convolution of the N pdf's $g_i(u)$, it is clear that the heuristic classifies the N-fold convolution as SLC.

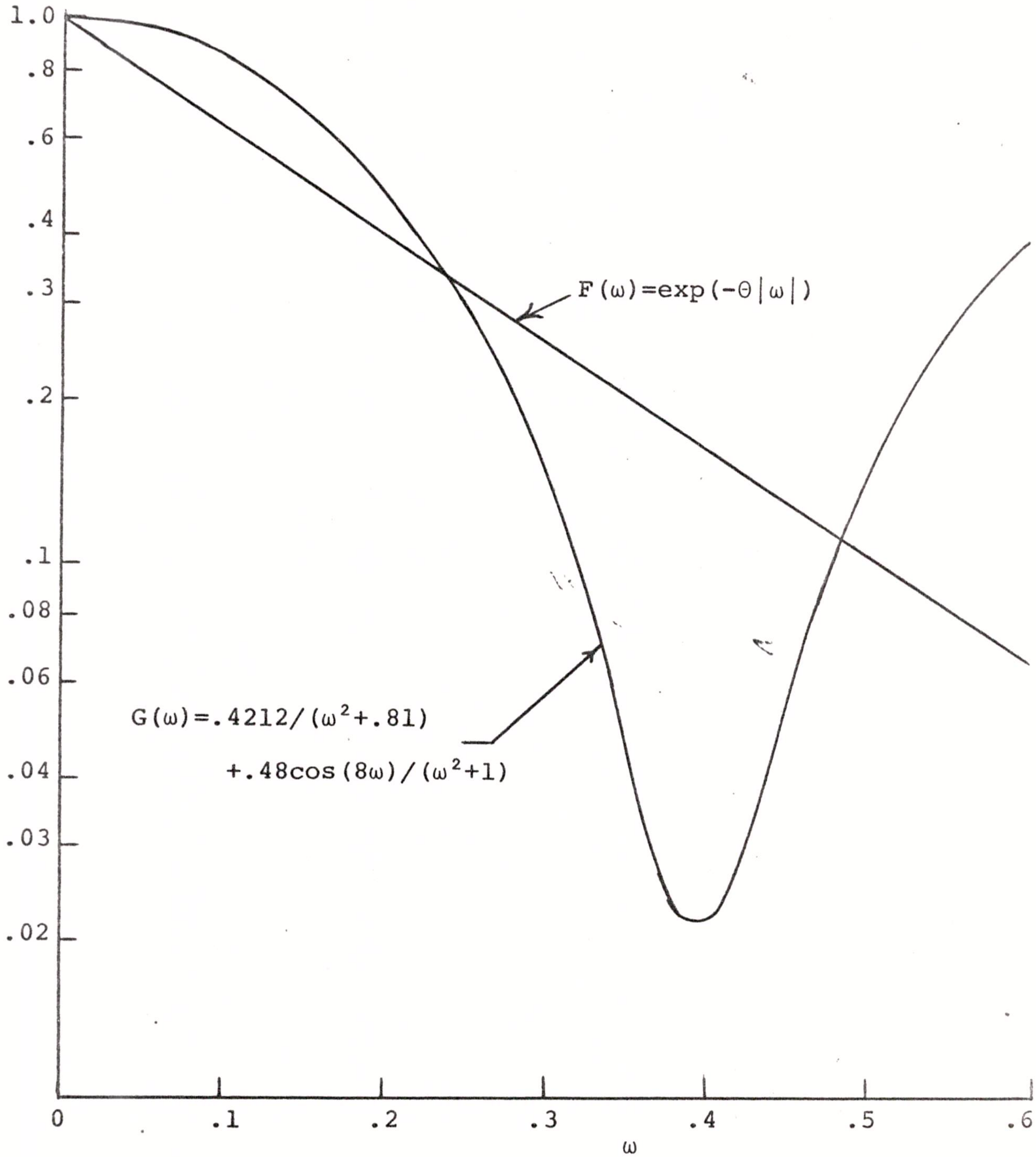
The converse is proven by reversing the sense of the inequalities in the first part of the proof. <u>QED</u>

1. HEURISTIC CLASSIFICATION OF AN ARBITRARY TRANSFORM $G(\omega)$[a]



[a] $F(\omega)$ is a standard function $\exp(-\Theta|\omega|)$ where Θ is chosen so that the intersection of the two curves occurs at $\omega_I=\pi/2a$. In (a), $G(\omega)$ is not self-linearly conformal. In (b), $G(\omega)$ is self-linearly conformal.

2. EXAMPLE APPLICATION OF THE HEURISTIC CLASSIFICATION METHOD TO CONDITIONALLY, LINEAR CONFORMAL PROBABILITY DISTRIBUTION[b]



[b]NOTE: The ordinate scale is logarithmic. The standard function $F(\omega)$ is therefore a straight line.

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