University of Pennsylvania

THE MOORE SCHOOL OF ELECTRICAL ENGINEERING

"INITIAL CONDITIONS" ALGORITHM FOR STATE VARIABLE ANALYSIS OF LINEAR SYSTEM

ABSTRACT

The marked increase in availability of digital computers has spread the use of state variable techniques for analysis of linear systems. In general, the initial conditions defined just before the input is applied differ in value from those defined just after the time of application. Since most time domain techniques require the "time greater than zero" initial conditions to be known, it is desired to be able to obtain the values of the initial conditions corresponding to one side of the origin from the values at the other side.

The initial-value property of the Laplace transform is found to be the key point in treating the problem. Algorithms based on this are developed using the differential equation, transfer function of the system, or the state variable assignment for the system. These algorithms provide programmable techniques for modifying the initial conditions to account for singularities at the origin. These methods are then extended to linear systems with delay, multiple inputs, and multiple outputs. Computer print-outs for several illustrative examples are included.

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Frederic Herman
AUTHOR

S D Bedrosian
FACULTY SUPERVISOR

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1. INTRODUCTION

This study is confined to systems described by linear differential equations. One way of solving a linear differential equation is directly in the time domain. In the state variable technique an nth-order differential equation is first converted to n first-order differential equations. Then numerical integration is used to determine the transient response corresponding to the system's input.

A problem arises when the initial values of the state variables are to be determined. One usually specifies the initial conditions (I.C.) of the system. The I.C. are defined at a time just previous to the application of the forcing function to the system. The input is therefore discontinuous at the time of application or origin, and it is therefore possible that the output of the system and/or its derivatives will also be discontinuous at the origin.

The various numerical techniques require that the I.C. of the system be known and that the solution or output of the system and its derivatives be continuous functions of time. Clearly, one must find the state of the system just after the forcing function has been applied, and then consider this state to be the initial state. Once this modification of the I.C. is performed, the transient response of the system can be correctly determined by numerical integration.

For simple circuit problems, the modified I.C. can usually be determined by inspection of the circuit. For a linear, fixed system described by its transfer functions rather than in terms of the ideal R-L-C components, it is not readily apparent how to modify the I.C. It is the purpose of this thesis to develop specific techniques for modifying the I.C. The desired techniques are of algorithmic form to facilitate their programming on a digital computer.

An alternative approach to solving linear differential equations is in frequency domain by means of Laplace transforms. Since the Laplace transform can account for discontinuities in a function and its derivatives, it is logical to approach the problem of modifying I.C. through a study of the Laplace transform. Before the derivation of the desired algorithms is attempted, it is instructive to consider the significance of the I.C.

2. INITIAL CONDITIONS CONCEPTS

In most engineering problems, a system is described in terms of differential equations. Often the transient response of the system to a given forcing function is of interest. To determine the response uniquely, one must be given additional information about the system not specified within the differential equations or the forcing functions.

Consider the single input, single output system of lumped components:

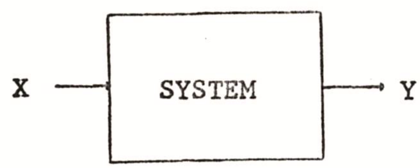

Figure 1

Block Diagram of a Single Input, Single Output System

If furthermore the system is linear with no time delays, the ordinary differential equation describing the system can always be written in the form:

$$\frac{d^n}{dt^n} y + a_{n-1}(t) \frac{d^{n-1}}{dt^{n-1}} y + a_{n-2}(t) \frac{d^{n-2}}{dt^{n-2}} y + \ldots + a_o(t) y =$$

$$b_m(t) \frac{d^m}{dt^m} x + b_{m-1}(t) \frac{d^{m-1}}{dt^{m-1}} x + \ldots + b_o(t) x \qquad (1)$$

where $n > m$ for a physical system. From the theory of differential equations, the general solution of Equation (1) is the sum of the complementary solution and the particular solution. The complementary solution can be expressed as:

$$y_c(t) = \sum_{k=1}^{n} c_k u_k(t) \tag{2}$$

where $u_1(t)$, $u_2(t)$, . . . $u_n(t)$ are n linearly independent solutions of the homogeneous equation associated with Equation (1), and where c_1, c_2, . . . c_n are arbitrary constants. It is apparent that one additionally needs n independent constraints on the system if he is to determine uniquely the transient response. If the values of y(t) and its first n-1 derivatives are known at some non-negative value of time, the n arbitrary constants of the complementary solution can be evaluated using these values and the expression for the general solution.

The transient response is usually thought of as beginning at some time, T, when the input to the system is initially applied. By a change of variables, the initial point can always be made to be the time t=0. The values $y(0)$, $y^{(1)}(0)$, . . . , $y^{(n-1)}(0)$ are called the "initial conditions" of the system. Except for the zero-input case, some function of time is suddenly applied to the system. The input is thus a singularity function. A singularity function and all its derivatives are continuous functions of time for all real values of time except possibly one[1]. (For this analysis, a piecewise continuous input can also be included since only the region near t=0 is of interest.) Since a linear system is only capable of

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(1) Reference 1, page 11.

the operations of amplification, integration, and differentiation, it is immediately seen that the output must also be a singularity function. The initial conditions must be defined in a limit, either as an identically negative t approaches zero from the negative values of time or as an identically positive t approaches zero from the positive values. As a matter of convenience, we will designate this first limit as the negative limit and the second limit as the positive limit. Also we shall, for convenience, designate the initial conditions defined in the sense of the negative limit as $I.C._-$ and that defined in the sense of the positive limit as $I.C._+$.

To illustrate the possibility of the I.C. being discontinuous, consider the circuit:

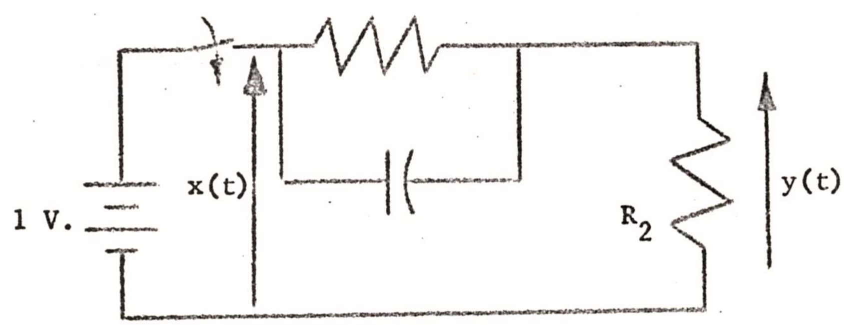

Figure 2

Simple R-C Circuit to Demonstrate Discontinuous Initial Conditions

where the capacitor is uncharged prior to the closure of the switch and $y(0_-) = 0$. When the switch closes, the capacitor is effectively a short circuit for that instant and the voltage across R_2 instantaneously jumps from zero to 1 volt. One can see that the value of $I.C._-$ is not the same as $I.C._+$.

For a set of equations describing a system, one may approach the solution in the time domain. Analytically, this usually means finding the particular solution. The I.C. are used to determine the arbitrary constants of the complementary solution. It is possible to use the I.C.$_-$, but rarely does an engineering text adopt this convention. The convention of the positive limit is preferred because of the difficulty of obtaining the particular solution for an input that is a singularity function.

For high order systems, the characteristic equation associated with the system cannot be factored except by numerical techniques, and then in only some cases. Solution by numerical integration is often used, especially when a digital computer is available. Since the usual methods of numerical integration cannot handle a singularity function directly except in limited cases, it becomes clear that in such cases the I.C.$_+$ must be found separately.

The Laplace transform is most versatile in handling the positive and negative limit conventions. Either convention can be adopted, provided that it is used consistently throughout a given problem. Properties such as the initial-value property and the time differentiation property must be used with respect to the convention adopted. The Laplace transform is considered in greater depth in the next section.

It becomes clear that the standard techniques for obtaining the transient response require the I.C.$_+$, with the exception of the Laplace transform defined through the negative limit. Unfortunately, in most types of problems involving physical systems, only the I.C.$_-$

are at one's disposal. For simple systems, it may be apparent how to find the I.C.$_{+}$ as in the circuit of Figure 2. As the complexity of the system increases, a method for obtaining the desired I.C. is no longer obvious.

Very often the transient response is desired for the case where the system was at rest previous to the initial point. Another problem of interest assumes that a system has reached a steady-state condition for one given input. Some other signal, perhaps noise, suddenly enters the system, and the response to the disturbance is desired. In both cases, the I.C.$_{-}$ are immediately known while I.C.$_{+}$ may be needed for the solution of the response.

In a circuit problem, the I.C.$_{-}$ are specified indirectly, in terms of initial voltages across capacitors and currents through inductors. Solving for the values of the output and its derivatives at the negative limit is nearly as tedious as solving directly for the I.C.$_{+}$. In this and certain other types of problems, it is convenient to use the state variable techniques since the state variables can be taken as the voltages across ideal capacitors and the currents in ideal inductors. The negative limit values of the state variables correspond to the I.C. specified in the problem. Unless the negative limit version of the Laplace transform is used to obtain the transient response, the problem of how to obtain I.C.$_{+}$ again presents itself.

In control systems analysis, the system is often described solely by a block diagram, where each block is given in terms of a transfer function. One cannot intuitively determine the I.C.$_{+}$ only by looking at the block diagram and a knowledge of the negative limit state of the system.

To solve the problem of finding the $I.C._{+}$, a procedure in the form of algorithms is desired. Not only can singularity function inputs be handled more easily, but inputs having several singularity points can also be handled.

3. LAPLACE TRANSFORM CONCEPTS

3.1 Introduction

In this section, some basic properties of the Laplace transform are considered with respect to positive and negative limits defined in the previous section. The "initial-value property" is of particular interest and it is extended to handle situations where it is normally considered to break down.

3.2 Singularities and the One-Sided Laplace Transform

The one-sided Laplace transform of a function f(t) defined for positive values of t, is defined as a function of s, by the integral

$$L\left\{f(t)\right\} \equiv \int_{0}^{\infty} f(t)e^{-st}dt \tag{3}$$

over the range of values of s for which the integral exists[2]. Sufficient but not necessary conditions for the existence of the Laplace transform of f(t) are[3]:

a. f(t) is at least piecewise continuous in every positive interval $t_1 \leq t \leq T$.

b. $t^n|f(t)|$ is bounded near t=0 for some number n, where $n < 1$.

c. There is a number s_o for which $e^{-s_o t}|f(t)|$ is bounded for large values of t.

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(2) Reference 3, page 53.

(3) Ibid, page 55.

Singularity functions such as the Dirac delta function $\delta(t)$ and its derivatives do not conform to the second condition stated above. In fact these functions are not considered true functions. They can nevertheless be handled by the Laplace transform as they arise in connection with many physical problems.

It is a matter of convention whether the lower limit of Equation (3) is taken as the negative limit 0_- or the positive 0_+. If f(t) is non-singular at the origin, the two conventions are equivalent. To see this consider Equation (3). Let the Laplace transform of f(t) defined by the negative limit convention be written as

$$L_- \left\{f(t)\right\} = F_-(s)$$

and the transform defined by the positive limit convention as

$$L_+ \left\{f(t)\right\} = F_+(s) \ .$$

Then one immediately sees that

$$L_- \left\{f(t)\right\} = \int_{0_-}^{\infty} f(t)e^{-st}dt = \int_{0_-}^{0_+} f(t)e^{-st}dt + L_+ \left\{f(t)\right\} .$$

If f(t) is first assumed to be continuous at the origin, then $f(t) = f(0)$ on the interval $0_- \leq t \leq 0_+$. Then

$$L_- \left\{f(t)\right\} = L_+ \left\{f(t)\right\} + f(0) \int_{0^-}^{0^+} e^{-st}dt = L_+ \left\{f(t)\right\}$$

for f(t) continuous and non-singular at t=0. If f(t) has a finite discontinuity, the same result can be obtained. For this case, assuming f(t) is still not singular at the origin, the function is bounded as $|f(t)| \leq k > 0$ on the interval $0_- \leq t \leq 0_+$. Then

$$\left| \int_{0_-}^{0_+} f(t)e^{-st}dt \right| \leq \int_{0_-}^{0_+} |f(t)| \, |e^{-st}| dt \leq k \int_{0_-}^{0_+} e^{-st}dt \leq (k)(0) = 0$$

which proves:

<u>Theorem 1:</u> If f(t) is non singular at the origin and is Laplace transformable, then $L_-\{f(t)\} = L_+\{f(t)\}$.

If singularities exist at the origin, the Laplace transform of f(t) will differ for the two limit conventions, since f(t) is either defined to contain the singularity or to be zero for $t \leq 0$ and be continuous for $t > 0$.

It is convenient at this point to consider the negative limit convention. From Theorem 1, there is no difference in the Laplace transform of regular functions found in the usual Laplace transform tables. It is further noted that properties of the Laplace transform such as the final-value theorem which do not explicitly use the values of f(t) or its derivatives at t=0 are equivalent for the two conventions. The only properties to consider are those of time differentiation and the initial-value theorem.

From Equation (3) we have

$$L_-\left\{\frac{df(t)}{dt}\right\} = \int_{0_-}^{\infty} \frac{d}{dt}\left[f(t)\right]e^{-st}dt .$$

Integrating by parts

$$\int_{0_-}^{\infty} \frac{df(t)}{dt} e^{-st}dt = f(t)e^{-st}\Big|_{0_-}^{\infty} + \int_{0_-}^{\infty} f(t)e^{-st}dt = f(0_-) + L_-\left\{f(t)\right\}$$

providing $L_-\left\{f(t)\right\}$ exists. The time differentiation property can be extended to higher order derivatives in the same way it is extended when the positive-limit convention is used. Thus[4]

$$L_-\left\{\frac{d^k}{dt^k}\left[f(t)\right]\right\} = s^k L_-\left\{f(t)\right\} - s^{k-1} f(0_-) - s^{k-2} f^{(1)}(0_-) \dots$$

$$\dots f^{(k-1)}(0_-). \tag{4}$$

The initial-value theorem is an extremely useful tool. If f(t) is non-singular at the origin, then regardless of the limit convention used[5]

$$\lim_{s \to \infty} s\, F(s) = f(0_+) .$$

To show this for the negative-limit case, consider Equation (4) for the value k=1. Thus

$$L_-\left\{f'(t)\right\} = \int_{0_-}^{\infty} e^{-st} f'(t)dt = s\, F_-(s) - f(0_-)$$

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(4) Reference 5, pages 176-179.

(5) Reference 5, page 181.

Taking the limit as $s \to \infty$

$$\lim_{s \to \infty} s F_-(s) - f(0_-) = \lim_{s \to \infty} \int_{0_-}^{\infty} e^{-st} f'(t)dt \tag{5}$$

Let $f'(t) = f'c(t) + f'd(t) + f's(t)$ where $f'c(t)$ is the continuous part of the derivative $f'd(t)$ is the discontinuous part, and $f's(t)$ is the singular part. The limit of the integral of the first two parts of the derivative multiplied by the exponential are clearly zero. We may represent the singular part as

$$f's(t) = \Delta_o \, \delta(t)$$

since $f(t)$ is assumed non-singular. Therefore, Equation (5) becomes

$$\lim_{s \to \infty} s F_-(s) - f(0_-) = \lim_{s \to \infty} \int_{0_-}^{\infty} e^{-st} \Delta_o \, \delta(t)dt.$$

Using[6] $\int_a^b g(t) \, \delta(t-c)dt = g(c)$

$$\lim_{s \to \infty} s F_-(s) - f(0_-) = \Delta_o \, .$$

But $f(0_+) - f(0_-) = \int_{0_-}^{0_+} \frac{df(t)}{dt} dt = \int_{0_-}^{0_+} \Delta_o \, \delta(t)dt = \Delta_o$

Therefore $\lim_{s \to \infty} s F_-(s) - f(0_-) = f(0_+) - f(0_-)$

$$\lim_{s \to \infty} s F_-(s) = f(0_+) \tag{6}$$

(6) Reference 4, page 55.

If the function f(t) is singular at the origin, then

$$\lim_{s \to \infty} s\,F(s) = f(t_o) \qquad (7)$$

does not exist[7]. Textbooks and technical papers very often abandon the initial-value property at this point and simply state that the property does not hold. Since impulses and their derivatives are Laplace transformable, the initial-value property can be extended in many cases to handle singular functions.

3.3 The Modified Initial Value Theorem

Suppose the Laplace transform F(s) of a function f(t) is known and suppose the limit of Equation (7) does not exist. It would seem that the function f(t) is singular at the origin. In many instances, information regarding the initial point can be obtained even if the function is singular.

If F(s) can be put in the form

$$F(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_o}{s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_o}, \qquad (8)$$

use of the initial-value theorem yields

$$f(0_+) = \lim_{s \to \infty} s\,F(s) = \lim_{s \to \infty} \left[b_m s^{m+1-n} \right] \qquad (9)$$

Then

$$f(0_+) = \begin{cases} 0 & m < n-1 \\ b_m & m = n-1 \\ \text{undefined} & m \geq n \end{cases}$$

(7) Reference 5, page 181.

The conventional interpretation of the initial-value property can only be used for the first two cases. Suppose m=n. Dividing the numerator by the denominator in Equation (8) yields

$$F(s) = b_m + \frac{(b_{m-1} - b_m a_{m-1})s^{m-1} + \ldots + (b_1 - b_m a_1)s + (b_o - b_m a_o)}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_o}$$

$$= b_m + F_1(s) \; . \qquad (10)$$

Then $f(t) = b_m \, \delta(t) + L^{-1}\left\{F_1(s)\right\}$.

If the initial-value theorem is applied to $F_1(s)$, one finds that $f_1(0_+) = (b_{m-1} - b_m a_{m-1})$. In terms of the function f(t), it is apparent that there is an impulse of magnitude b_m at the origin while the value of f(t) a small time after the impulse is given by $f_1(0_+)$.

Is $m > n$, then $F_1(s)$ would also be singular at the origin. By repeated division of the numerator of the fractional term in F(s), by the denominator, the remainder term is reduced until the order of the numerator is less than the order of the denominator. The fraction can then be handled by the initial-value theorem while the terms of powers of s can be considered to represent impulses of order equal to the corresponding powers of s. An algorithm to reduce Equation (8) to

$$F(s) = c_o s^{m-n} + c_1 s^{m-n-1} + \ldots + c_k s^{m-n-k} + \ldots + c_{m-n} +$$

$$\frac{d_{n-1}s^{n-1} + d_{n-2}s^{n-2} + \ldots + d_1 s + d_o}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_o} \qquad (11)$$

is derived in Appendix 1. The constants found by the algorithm[8] are

$$c_o = b_m \tag{12}$$

$$c_k = b_{m-k} - \sum_{i=1}^{k} c_{k-i}\, a_{n-i} \; ; \quad k = 1,2, \ldots , m-n$$

$$d_i = b_i - \sum_{j=o}^{m-n} c_j\, a_{n-m+i+j} \; ; \quad i = 0,1, \ldots , n-1 \tag{13}$$

where $a_r=0$ if $r < 0$.

It is clear that if the inverse Laplace transform is found, the function $f(t)$ will have impulses at the origin of:

$$\begin{array}{l} \text{order } m-n \text{ and magnitude } c_o \\ \text{order } m-n-1 \text{ and magnitude } c_1 \\ \cdots\cdots\cdots\cdots \\ \text{order } 1 \text{ and magnitude } c_{m-n-1} \\ \text{order } 0 \text{ and magnitude } c_{m-n} \end{array} \tag{14}$$

where an n^{th} order impulse is

$$\frac{d^n}{dt^n}\left[\delta(t)\right] .$$

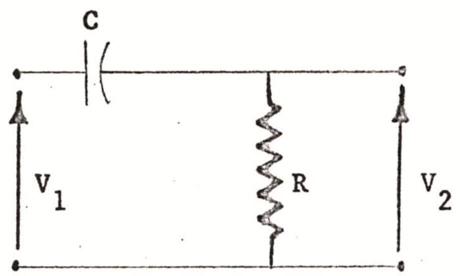
The value of the function a differential time $\epsilon > 0$ after $t=0$ is

$$d_{n-1} = b_{n-1} - \sum_{j=0}^{m-n} c_j\, a_{2n+j-m-1} \tag{15}$$

where $a_r = 0$ for $r < 0$ and $c_j = 0$ for $r < 0$.

(8) Algorithm is only used when $m \geq n$.

In many cases, the impulses existing at the origin are in themselves of little interest. When the output of the system is the input of another system, an impulse or its derivative may be significant. When susceptability of a digital system is a critical concern, one is interested in the effect of high order impulses. In physical systems, the transfer function does not have a higher order numerator than denominator so that it cannot generate an impulse of order higher than impulses in the input. However, in control applications, derivative control is approximately realizable such as in Figure 3.



$$\frac{V_2(s)}{V_1(s)} = \frac{RCs}{RCs + 1}$$

Figure 3

A Simple Differentiation Circuit

If $RC << 1$, then the circuit is approximately a differentiator and the output will be approximately an impulse for a step input. It is therefore possible to generate impulsive signals somewhere in a system even though the input is continuous.

In most cases, however, one does not need to consider impulses at the origin. For practical purposes, the initial point is taken just after the origin. Additionally one may not be interested in the possibility of the derivatives of the output containing impulses at the origin as long as the transient response obtained is valid for $t > 0$. On this assumption we state:

Theorem 2: Modified Initial-Value Theorem:

If a. the Laplaqe transform of f(t) exists;

b. $\lim_{s \to \infty} \left[s\,F(s) \right]$ does not exist;

c. F(s) can be put in the form of Equation (8);

then $f(\epsilon)$, where ϵ is the positive limit point of the origin, is given by Equation (15), where $c_o, c_1, \ldots, c_{m-n}$ are determined by Equation (12) and $f(\epsilon) \equiv f(0_+)$ is defined as the "modified initial-value" of f(t). Impulses at the origin are given by Equation (14).

If F(s) cannot be put in the form of Equation (8) such as in the case of delay paths in the system, the problem is complicated. Several cases of these functions will be considered in later sections.

It is ultimately desired to be able to determine the modified initial-values of the n-1 derivatives of f(t) for an n^{th} order system. Using Equation (6) and letting $F(s) = L_- \left\{ \frac{d^k}{dt^k} f(t) \right\}$

$$f^{(k)}(0_+) = \lim_{s \to \infty} s\left[L_- \left\{ \frac{d^k}{dt^k} f(t) \right\} \right]. \tag{16}$$

Now using Equation (4) in Equation (16) yields

$$f^{(k)}(0_+) = \lim_{s \to \infty} \left[s^{k+1}F_-(s) - s^k f(0_-) - s^{k-1} f^{(1)}(0_-) - \ldots - s^2 f^{(k-2)}(0_-) - s\, f^{(k-1)}(0_-) \right] \tag{17}$$

It may appear that the terms of Equation (17) which contain the I.C.$_-$ generate impulses. But, in fact, they cancel with terms produced by $s^{k+1}F_-(s)$. To show this, consider a simplified equation

$$\frac{d^n}{dt^n} y(t) = x(t) \tag{18}$$

Using Equation (4), Equation (18) becomes

$$s^n Y(s) = X(s) + s^{n-1} y(0_-) + s^{n-2} y^{(1)}(0_-) + \ldots + s\, y^{(n-2)}(0_-) + y^{(n-1)}(0_-)$$

$$Y(s) = \frac{X(s)}{s^n} + \frac{y(0_-)}{s} + \frac{y^{(1)}(0_-)}{s^2} + \ldots + \frac{y^{(n-2)}(0_-)}{s^{n-1}} + \frac{y^{(n-1)}(0_-)}{s^n}$$

Then using Equation (16) for the k^{th} derivative, $k < n$

$$y^{(k)}(0_+) = \lim_{s \to \infty} \left[\frac{X(s)}{s^{n-k-1}} + s^k y(0_-) + s^{k-1} y(0_-) + \ldots \right.$$

$$+ s\, y^{(k-1)}(0_-) + y^k(0_-) + \frac{y^{(k+1)}(0_-)}{s} + \ldots + \frac{y^{(n-1)}(0_-)}{s^{n-k-1}}$$

$$\left. - s^k y(0_-) - s^{k-1} y^{(1)}(0_-) - \ldots - s^2 y^{(k-2)}(0_-) - s\, y^{k-1}(0_-) \right]$$

Cancelling terms we obtain

$$y^{(k)}(0_+) = \lim_{s \to \infty} \left[\frac{X(s)}{s^{n-k-1}} + y^{(k)}(0_-) + \frac{y^{(k-1)}(0_-)}{s} + \ldots + \frac{y^{(n-1)}(0_-)}{s^{n-k-1}} \right]$$

$$y^{(k)}(0_+) = \lim_{s \to \infty} \left[\frac{X(s)}{s^{n-k-1}} \right] + y^{(k)}(0_-) \tag{19}$$

provided that the limit can be evaluated.

A more general system will yield results similar to Equation (19). Intuitively one sees that this must be so. If there is no input, it is clear that the only term on the right-hand side of Equation (19) will be $y^k(0_-)$ since there is no discontinuity in the input. Similarly, if the I.C.$_-$ are zero, the only contribution to the initial values at $t=0_+$ must be due solely to the input.

It is now obvious how the I.C.$_+$ are determined from I.C.$_-$. If the system of Figure 1 is described by

$$\frac{d^n}{dt^n} y(t) + a_{n-1} \frac{d^{n-1}}{dt^{n-1}} y(t) + \ldots + a_o y(t) =$$

$$b_m \frac{d^m}{dt^m} x(t) + b_{m-1} \frac{d^{m-1}}{dt^{m-1}} x(t) + \ldots + b_o x(t) \tag{20}$$

Then the transformed equation is

$$(s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_o)\, Y(s) =$$

$$(b_m s^m + b_{m-1}s^{m-1} + \ldots b_o)\, X(s) + \text{I.C.}_- \text{ terms}$$

$$Y(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_o}{s^n + a_{n-1}s^{n-1} + \ldots + a_o} X(s) + \frac{\text{I.C. terms}}{s^n + a_{n-1}s^{n-1} + \ldots + a_o} \tag{21}$$

From the above discussion, Theorem 2, and Equation (19), we obtain[9]

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(9) This result is obtained in Reference 8, page 175, except for the limited being a conventional limit. The requirement that this limit exist is typical of the treatment of the initial-value theorem in the literature. The special limit of Equation (22) usually makes possible the extention of the initial-value theorem to functions that are singular at the origin.

$$y^{(k)}(0_+) = y^{(k)}(0_-) + \lim^*_{s \to \infty} \left[s^{k+1} \frac{(b_m s^m + b_{m-1} s^{m-1} + \ldots + b_o)}{(s^n + a_{n-1} s^{n-1} + \ldots + a_o)} X(s) \right] \quad (22)$$

where $\lim^*_{s \to \infty}$ denotes that this is a special limit. This is to be evaluated as a regular limit, if this limit exists, or else to be evaluated using Theorem 2 or the techniques that Theorem 2 embodies.

In addition to the change

$$y^{(k)}(0_+) - y^{(k)}(0_-) = \Delta\, y^{(k)}$$

the impulses of $y^k(t)$ at the origin are also determined using the algorithm defined by Equation (12) and Equation (13).

It is also possible to determine the I.C.$_-$ from a knowledge of I.C.$_+$ and the input function using Equation (22), although one is rarely interested in working in this direction.

Algorithms that yield the initial value information for the output and its derivatives are developed in the next section for the generalized function F(s) and then for some specific inputs.

4. ALGORITHMS TO MODIFY INITIAL CONDITIONS FOR A LINEAR, FIXED, SINGLE INPUT/OUTPUT SYSTEM

If the system of Figure 1 is linear, fixed, and has no delay paths, the differential Equation (20) may be used to describe the system. The transform of the output is given by Equation (21) which is repeated below.

$$Y(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_o}{s^n + a_{n-1} s^{n-1} + \ldots + a_o} X(s) + \frac{\text{I.C. terms}}{s^n + a_{n-1} s^{n-1} + \ldots a_o} \tag{23}$$

Inspection of a table of Laplace transforms[10] indicates that the transform of virtually any function that is of practical interest as an input can be written as the quotient of two polynomials in s[11]. Assuming this condition, Equation (23) is then written as

$$Y(s) = \frac{B_p s^p + B_{p-1} s^{p-1} + \ldots + B_o}{s^q + A_{q-1} s^{q-1} + \ldots + A_o} + \frac{\text{I.C. terms}}{s^n + a_{n-1} s^{n-1} + \ldots + a_o} \tag{24}$$

Using Equation (22) we write the initial conditions as

$$y^{(k)}(0_+) = y^{(k)}(0_-) + \lim_{s \to \infty}{}^{*} \left[s^{k+1} \left(\frac{B_p s^p + B_{p-1} s^{p-1} + \ldots + B_o}{s^q + A_{q-1} s^{q-1} + \ldots + A_o} \right) \right] \tag{25}$$

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(10) Reference 6, Pages 328-335.

(11) The method of handling functions that do not conform to this requirement is considered in the last part of this section.

This n^{th} order system requires $y(0_+)$, $y^{(1)}(0_+)$, . . . , $y^{(n-1)}(0_+)$ as the $I.C._+$. Additionally, one might require that any impulses present at the origin of $y(t)$, $y^{(1)}(t)$, . . . $y^{(n-1)}(t)$ be determined. Let

$$F(s) = \frac{B_p s^p + B_{p-1} s^{p-1} + \ldots + B_o}{s^q + A_{q-1} s^{q-1} + \ldots + A_o} \tag{26}$$

Using the expansion algorithm of Equation (12), Equation (26) becomes

$$F(s) = c_o s^{p-q} + c_1 s^{p-q-1} + \ldots + c_{p-q} + \frac{c_{p-q+1}}{s} + \ldots$$

$$+ \frac{c_{p+n-q}}{s^n} + \frac{d_{q-n-1} s^{q-n-1} + d_{q-n-2} s^{q-n-2} + \ldots + d_{-n} s^{-n}}{s^q + A_{q-1} s^{q-1} + \ldots + A_o} \tag{27}$$

where $c_o = B_p$

$$c_j = B_{p-j} - \sum_{i=1}^{j} c_{j-i} A_{q-i} \; ; \quad j = 1, 2, \ldots, p+n-q$$

$$A_r = 0 \text{ for all } r < 0$$

Using Equation (27), the initial conditions expression (25) becomes

$$y^{(k)}(0_+) = y^{(k)}(0_-) + \lim_{s \to \infty}{}^{*} \left[s^{k+1} \left(\sum_{j=0}^{p+n-q} c_j s^{p-q-j} + \right.\right.$$

$$\left.\left. \frac{d_{q-n-1} s^{q-n-1} + d_{q-n-2} s^{q-n-2} + \ldots + d_{-n} s^{-n}}{s^q + A_{q-1} s^{q-1} + \ldots + A_o} \right) \right] \tag{28}$$

where $k = 0, 1, \ldots, n-1$

One notices that

$$\lim_{s \to \infty} s^{k+1} \left[\frac{d_{q-n-1}s^{q-n-1} + d_{q-n-2}s^{q-n-2} + \dots + d_{-n}s^{-n}}{s^q + A_{q-1}s^{q-1} + \dots + A_o} \right]$$

exists and equals zero for $(k = 0,1, \dots, n-1)$. Then Equation (28) simplifies to

$$y^{(k)}(0_+) = y^{(k)}(0_-) + \lim_{s \to \infty}{}^{*} \sum_{j=0}^{p-q+n} (c_j s^{p+k+1-j-q}) \tag{29}$$

From the definition of $\lim^*_{s \to \infty}$ given by Theorem 2, and from the discussion leading to the theorem, one clearly sees that

$$\lim_{s \to \infty}{}^{*} s(s^{r-1}) = \lim_{s \to \infty}{}^{*}(s^r) = \begin{cases} 0 & , r < 0 \\ 1 & , r = 0 \\ 0 & , r > 0 \end{cases}$$

so that Equation (29) becomes

$$y^{(k)}(0_+) = y^{(k)}(0_-) + c_{p+k+1-q} \tag{30}$$

where $c_o = B_p$

$$c_j = B_{p-j} - \sum_{i=1}^{j} c_{j-i}A_{q-i} \; ; \quad j = 1,2, \dots, p+n-q$$

$$A_r = 0 \text{ for all } r < 0$$

If the limit of Equation (29) is written as

$$\lim_{s \to \infty}{}^{*} s\left[\sum_{j=0}^{p+n-q} c_j s^{p+k-j-q} \right] = \lim_{s \to \infty}{}^{*} s\, G(s) ,$$

the impulses at the origin are determined as was done in the development of Equation (14) by considering the inverse of G(s). Thus one immediately sees that for $p+k-q < 0$, no impulses occur in $y^{(k)}(t)$ at the origin. Otherwise the impulses of $y^{(k)}(t)$ are of

order	and	magnitude
$p+k-q$		c_o
$p+k-q-1$		c_1
.		.
.		.
.		.
$p+k-j-q$		c_j
.		.
.		.
.		.
1		$c_{p+k-q-1}$
0		c_{p+k-q}

(31)

If the output of a system can be represented by Equation (24), the $I.C._+$ can be found from Equation (30) and the $I.C._-$. The impulses of the output and its n-1 derivatives at the origin are given by Equation (31). While one can easily multiply the Laplace transform of the input times the transfer function, it is desired to derive algorithms that will directly yield the initial value information without explicit use of the Laplace transform. Such algorithms are derived in Appendix II for some standard input functions. The algorithms are derived by actually multiplying the Laplace transform of the input by the general transfer function,

$$T(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_o}{s^n + a_{n-1} s^{n-1} + \ldots + a_o} \tag{32}$$

to obtain Equation (26). The constants of Equation (26) can then be related to constants in Equation (32) and in the expression for the input function transform. With these constants known, the algorithms are obtained by substituting these constants into Equations (30) and (31).

In Equations (30) and (31), generalized terms are used to put the expressions in a compact form. However, the generalized subscripts of a constant may sometimes take on a value outside the range for which the constant is defined. Usually such a constant can be interpreted to be zero. The single exception is for a_n which is taken as unity although a_n does not appear explicitly in Equation (23). When an index of a summation or recursive operation is initially past the final limit in the direction of the index change, this operation should be deleted until the summation or recursion is again initiated. For the derivation of the algorithms of Appendix II, this procedure has been used.

To facilitate application of the algorithms of Appendix II, one must have a transfer function in the form of Equation (32) or a differential equation of the form of Equation (20) to define the constants a_j, b_j. Table 1 summarizes the algorithms. The input function is found in the first column. J_m and α_1 are evaluated using the expressions in the same row as the input. If J_m is negative, the

<u>Table 1</u>

No.	Input	Function	J_m	α_1	c_j	$c_o = \alpha_1 b_m$
1	$\alpha\,\delta^{(L)}(t)$	L Order Impulse	m+L	α	$\alpha_1 b_{m-j} - \sum_{i=1}^{j} c_{j-i} a_{n-i}$	
2	$\alpha\, t^L$	Power of t	m-L-1	$\alpha[L!]$	$\alpha_1 b_{m-j} - \sum_{i=1}^{j} c_{j-i} a_{n-i}$	
3	$\alpha\, e^{\beta t}$	Exponential	m-1	α	$\alpha_1 b_{m-j} - \sum_{i=1}^{j} c_{j-i}\left[a_{n-i} - \beta a_{n+1-i}\right]$	
4	$\alpha \cosh \beta t$	Hyperbolic Cosine	m-1	α	$\alpha_1 b_{m-j} - \sum_{i=1}^{j} c_{j-i}\left[a_{n-i} - \beta^2 a_{n+2-i}\right]$	
5	$\alpha \cos \beta t$	Cosine	m-1	α	$\alpha_1 b_{m-j} - \sum_{i=1}^{j} c_{j-i}\left[a_{n-i} + \beta^2 a_{n+2-i}\right]$	
6	$\alpha\, e^{\beta t} \cos \gamma t$	Exponential Cosine	m-1	α	$\alpha_1\left[b_{m-j} - \beta\, b_{m+1-j}\right] - \sum_{i=1}^{j} c_{j-i}\left[a_{n-i} - 2\beta a_{n+1-i} + (\beta^2 + \gamma^2) a_{n+2-i}\right]$	

Table 1 (continued)

No.	Input	Function	J_m	α_1	c_j
					$c_o = \alpha_1 b_m$
7	$\alpha t \sin \beta t$	Time Sinusoid	m-3	$2\alpha\beta$	$\alpha_1 b_{m-j} - \sum_{i=1}^{j} c_{j-i}\left[a_{n-i} + 2\beta^2 a_{n+2-i} + \beta^4 a_{n+4-i}\right]$
8	$\alpha \sin \beta t$	Sinusoid	m-2	$\alpha\beta$	$\alpha_1 b_{m-j} - \sum_{i=1}^{j} c_{j-i}\left[a_{n-i} + \beta^2 a_{n+2-i}\right]$
9	$a \sinh \beta t$	Hyperbolic Sine	m-2	$\alpha\beta$	$\alpha_1 b_{m-j} - \sum_{i=1}^{j} c_{j-i}\left[a_{n-i} - \beta^2 a_{n+2-i}\right]$
10	$\alpha t e^{\beta t}$	Time Exponential	m-2	α	$\alpha_1 b_{m-j} - \sum_{i=1}^{j} c_{j-i}\left[a_{n-i} - 2\beta\, a_{n+1-i} + \beta^2 a_{n+2-i}\right]$
11	$\alpha e^{\beta t} \sin \gamma t$	Exponential Sinusoid	m-2	$\alpha\gamma$	$\alpha_1 b_{m-j} - \sum_{i=1}^{j} c_{j-i}\left[a_{n-i} - 2\beta\, a_{n+1-i} + (\beta^2 + \gamma^2) a_{n+2-i}\right]$
12	$\alpha t \cos \beta t$	Time Cosine	m-2	α	$\alpha_1\left[b_{m-j} - \beta^2 b_{m+2-j}\right] - \sum_{i=1}^{j} c_{j-i}\left[a_{n-i} + 2\beta^2 a_{n+2-i} + \beta^4 a_{n+4-i}\right]$

Note: $a_r, b_r = 0; \quad a_n = 1 \quad \{r < 0,\ r > n\}$

initial values do not have to be modified. If J_m is zero, ($c_o = \alpha_1 b_m$) is computed and the initial value

$$y^{(n-1)}(0_+) = y^{(n-1)}(0_-) + c_o$$

is found. For a positive J_m, one also evaluates c_j of the last column of Table 1 by letting j equal one through the value J_m. Having the constants c_j, one can then find the I.C.$_+$ using

$$y^{(k)}(0_+) = y^{(k)}(0_-) + c_{J_m+k-n+1} \quad ; \quad k = 0,1, \ldots, n-1$$

where $c_r = 0$ for all $r < 0$.

The impulses existing at the origin for the function $y^{(k)}(t)$ (where $k = 0,1, \ldots, n-1$) are of

order	J_m+k-n	and magnitude	c_o
	.		.
"	.	"	.
	.		.
"	1	"	$c_{J_m+k-n-1}$
"	0	"	c_{J_m+k-n}

Obviously, if $J_m+k-n < 0$, no impulses exist for $y^{(k)}(t)$.

Example

Given the transfer function

$$T(s) = \frac{5s^3 + 2s^2 + 3s + 2}{s^4 + 2s^3 + s^2 + s + 1}$$

the input

$$x(t) = e^{-3t} \sin 2t$$

and

$$y(0_-) = 1$$

$$y'(0_-) = 0$$

$$y''(0_-) = 0$$

$$y^{(3)}(0_-) = -1$$

find the "initial conditions" at $t = 0_+$.

We have

$$b_o = 2 \qquad a_o = 1 \qquad n = 4$$

$$b_1 = 3 \qquad a_1 = 1 \qquad m = 3$$

$$b_2 = 2 \qquad a_2 = 1$$

$$b_3 = 5 \qquad a_3 = 2$$

Using Table 1

$$J_m = 3-2 = 1 \qquad c_o = (2)(5) = 10$$

$$\alpha_1 = 1(2) = 2 \qquad c_1 = \alpha_1 b_2 - c_o \left[a_3 - (2)(-3)(1) + 0\right]$$

$$= (2)(2) - 10\ [2+6] = 4-80$$

$$= -76$$

Then

$$y^{(k)}(0_+) = y^{(k)}(0_-) + c_{k-2}$$

$$y(0_+) = 1$$

$$y'(0_+) = 0$$

$$y''(0_+) = 10$$

$$y^{(3)}(0_+) = -77$$

Finally, $y(t)$, $y'(t)$, and $y''(t)$ have no impulses at the origin while $y^{(3)}(t)$ has a zero order impulse of magnitude 10.

It is a matter of convenience whether to use the direct algorithms for specific inputs or to use the general algorithm of Equations (30) and (31) for the product of the transfer function and the input transform. It may be convenient to program the algorithms on a digital computer. In this case, it is probably simpler to use the general algorithm and have the program multiply the transfer function by the transform of the input. Computer programs of both methods have written in Fortran IV and are in Appendix III with a brief description.

The foregoing discussion assumed that the Laplace transform of the input could be written as the quotient of two polynomials in s. While for most practical problems, the input meets this condition, it is of interest to consider some functions which do not. One can examine Laplace transform tables[12] and intuitively see that the complete solution of the system for such an input would have to be determined numerically in most cases. A fractional power of t is such a function and it will have a transform with a fractional power of s. The product of this time function with the system transfer function will also have fractional powers of s. If the conventional initial-value theorem cannot be applied because the limit involved does not exist, the modified initial value theorem won't work either since division of the product function, numerator by denominator, will result in fractional powers of s. The output has negative fractional powers of t, and there is no limit point to the right of the origin

(12) Reference 6, pages 328-335.

for the output function.

In general, the modified initial-value theorem cannot be used unless the input function can be reduced to a quotient of two polynomials. Otherwise the I.C.$_+$ can be found only if

$$\lim_{s \to \infty} s^n \, T(s) \, X(s) \quad \text{exists}^{(13)},$$

where T(s) is the transfer function of an n^{th} order system and X(s) is the input. A possible way to reduce a function to a quotient of two polynomials is to expand all non-polynomial expressions within the function in power series. Usually, only a few terms of the series are needed for the purpose of applying the modified initial-value theorem.

Example

Given $F(s) = \tan^{-1}\left(\frac{a}{s} \right)$

$f(0_-) = 0$ find $f'(0_+)$.

Using Equation (17)

$$f'(0_+) = \lim_{s \to \infty} \left[s^2 \tan^{-1} \frac{a}{s} \right] \tag{33}$$

and the expansion(14)

$$\tan^{-1}\left(\frac{a}{s}\right) = \frac{a}{s} - \frac{1}{3}\frac{a^3}{s^3} + \frac{1}{5}\frac{a^5}{s^5} - \cdots$$

(13) Reference 8 provides some useful techniques for evaluation of indeterminant limits.

(14) Reference 6, page 373.

we get

$$f'(0_+) = \lim_{s \to \infty} s\left[a - \frac{1}{3}\frac{a^3}{s^2} + \frac{1}{5}\frac{a^5}{s^4} - \cdots\right]$$

Using the modified initial-value theorem

$f'(0_+) = 0$; 0 order impulse of magnitude a at the origin.

5. GENERALIZATION TO LINEAR SYSTEM

5.1 Introduction

In previous chapters, the systems considered were assumed to be:

1. Single input-single output
2. Linear
3. Time-invariant
4. Without delay paths

In addition to the requirement on the input that it be a single input related to the output by either a single differential equation or a single transfer function, the input was assumed to be a single function of time rather than the sum of several elementary functions.

While the above assumptions are valid for the usual systems-analysis problems, there are times when one or more of these assumptions do not apply. This chapter is concerned with extending the previously developed techniques to the more general type of linear system.

5.2 Composite Input Functions

System analysis frequently makes use of elementary functions such as the step and the ramp functions. In Chapter 4, algorithms for specific input functions were developed. If the input function is the sum of several functions, the assumption of linearity allows the superposition property to be used to simplify analysis of the system.

Suppose the input function is written as the sum of terms

$$x(t) = k_1x_1(t) + k_2x_2(t) + . . . + k_Lx_L(t) \quad (34)$$

where

$$x_1(t), x_2(t), . . . , x_L(t)$$

are elementary input functions, and the coefficients

$$k_1, k_2, . . . , k_L$$

are constants. Following Equation (23), denoting the system transfer function by T(s), the Laplace transform of the output is given by

$$Y(s) = T(s) \left[k_1X_1(s) + k_2X_2(s) + . . . + k_LX_L(s)\right] + \text{I.C. terms} \quad (35)$$

From Equation (35), the effect of the input is equivalent to the sum of effects of each term of the input. The system is not linear in the strictest sense since superposition does not hold except for the case of zero initial conditions. If the particular solution of the system is considered separately, one can apply the property of superposition. To obtain the $I.C._+$, the effect of each input term can be separately determined using Theorem 2 or appropriate algorithms. The net change in the I.C. is the sum of the separate changes due to each term of the input. If numerical integration techniques are employed, it may be convenient to compute the response due to each input term separately. The response due to the entire input function will then be the sum of the separate responses providing the effect of the $I.C._-$ is correctly included. One way to correctly account for the $I.C._-$ effects is to compute the individual responses for zero $I.C._-$ and then add the sum of these responses to the zero input response corresponding to the actual I.C._ . Another way is to compute the separate responses corresponding to zero I.C._ for all but one of the input terms. For this remaining term the values of the I.C._ are taken as the actual values. This second method combines the homogeneous solution with the particular solution for one of the input terms.

5.3 Multiple Input-Single Output Systems

If a system has several inputs not all applied to the system at the same place, the system is a multiple input system. If the system is additionally a linear, fixed, single-output system, the Laplace transform for the system output can be expressed as

$$Y(s) = T_1(s)\, X_1(s) + T_2(s)\, X_2(s) + \ldots + T_L(s)\, X_L(s) \tag{36}$$

where

$$X_1(s),\ X_2(s),\ \ldots,\ X_L(s)$$

are the Laplace transforms of the inputs, and

$$T_1(s),\ T_2(s),\ \ldots,\ T_L(s)$$

are the transfer functions defined by

$$T_i(s) \equiv \left.\frac{Y(s)}{X_i(s)}\right|_{\text{I.C.} = 0,\ X_j(s) = 0 \text{ for all } j \neq i}$$

If the transient response is to be obtained in the time domain, Equation (36) must be transformed into a differential equation. Assuming that all the transfer functions can be expressed as the ratio of two polynomials in s, Equation (36) can be expressed as

$$Y(s) = \frac{P_1(s)\ X_1(s) + P_2(s)\ X_2(s) + \ . \ . \ . + P_L(s)\ X_L(s)}{D(s)} + \text{I.C. terms} \quad (37)$$

where D(s) is a common denominator for the transfer functions. Unless D(s) is the L.C.D. (least common denominator), its order will be higher than that of the L.C.D. Dropping the I.C. terms, multiplying each side of Equation (37) by D(s), and then using the transformation

$$s^n \rightarrow D^p$$

where D^p is the differential operator

$$\frac{d^p}{dt^p}$$

one has a differential equation of order equal to that of D(s). The degree to which the denominators of the transfer functions can be factored determines how low an order D(s) can be found. If one is forced to a time domain solution it is usually because the denominators of the transfer functions cannot be factored, and the D(s) polynomial may therefore be the product of all the denominators of the transfer functions.

If the Modified Initial-Value Theorem is applied to Equation (36), for the j^{th} derivative of the output it is clear that

$$y^{(j)}(0_+) = y^{(j)}(0_-) + \sum_{i=1}^{L} \Delta_L^{(j)} \tag{38}$$

where

$$\Delta_i^{(j)} = \lim_{s \to \infty}{}^{*} \left[s^{j+1} T_i(s) X_i(s) \right]$$

The higher the order of the resulting differential equation, the greater the number of I.C.$_-$ must be specified for the system. There is a minimum number of independent conditions which will uniquely determine the state of a system and this so-called order of the system is independent of the particular representation of the system.[15]

When a system can be represented as a combination of ideal R-L-C elements, it is convenient to use state variables to represent the system. Usually the state variables can be chosen so that they are linearly independent quantities. Their initial values then represent the minimum number of quantities which characterize the initial state. State variables are considered later in this chapter.

5.4 Multiple-Output Systems

A multiloop system is usually first specified through n linearly independent simultaneous differential where n is the number of unknowns appearing in the equations. The equations are Laplace transformed so that the equations for zero I.C. are of the form

(15) Reference 5, page 52.

$$D_{11}Y_1(s) + D_{12}Y_2(s) + \ldots + D_{1N}Y_N(s) = B_{11}X_1(s) + B_{12}X_2(s) + \ldots + B_{1M}X_M(s)$$
$$D_{21}Y_1(s) + D_{22}Y_2(s) + \ldots + D_{2N}Y_N(s) = B_{21}X_1(s) + B_{22}X_2(s) + \ldots + B_{2M}X_M(s)$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \qquad (39)$$
$$D_{N1}Y_1(s) + D_{N2}Y_2(s) + \ldots + D_{NN}Y_N(s) = B_{N1}X_1(s) + B_{N2}X_2(s) + \ldots + B_{NM}X_M(s)$$

where D_{ij}, B_{ij} are polynomials in s. The corresponding matrix representation is

$$\begin{bmatrix} D_{11} & D_{12} & \cdots & D_{1N} \\ D_{21} & D_{22} & \cdots & D_{2N} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ D_{N1} & D_{N2} & \cdots & D_{NN} \end{bmatrix} \begin{bmatrix} Y_1(s) \\ Y_2(s) \\ \cdot \\ \cdot \\ \cdot \\ Y_N(s) \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1M} \\ B_{21} & B_{22} & \cdots & B_{2M} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ B_{N1} & B_{N2} & \cdots & B_{NM} \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \\ \cdot \\ \cdot \\ \cdot \\ X_M(s) \end{bmatrix} \qquad (40)$$

or

$$\underline{D}\ \underline{Y} = \underline{B}\ \underline{X}$$

Suppose that the highest order derivative for each output Y_i is equal to L_i. The highest possible order of this system will be

$$\sum_{i=1}^{N} L_i$$

and the determinant of the $\underline{D}$ matrix will be a polynomial of this order(16). If the inverse of $\underline{D}$ multiplies each side of Equation (40), the result is

(16) Reference 5, page 38.

$$Y(s) = \underline{D}^{-1}\underline{B}\ \underline{X}(s) \quad . \tag{41}$$

Each row of this matrix is of the form

$$Y_i(s) = \frac{P_{i1}X_1(s) + P_{i2}X_2(s) + \ldots + P_{iM}X_M(s)}{D(s)} \tag{42}$$

where

$$D(s) = \text{Det}\ [\underline{D}(s)] \quad .$$

Comparing Equation (42) with Equation (37), it is seen that the system is reduced to N uncoupled multiple input/single output systems. To obtain the response of each $Y_i(t)$, the initial value of $Y_i(t)$ and its

$$\left(\sum_{i=1}^{N} L_i \right) - 1$$

derivatives are needed. The modification of the I.C. and the transient solution is handled as discussed in the previous part of this chapter. If the response of more than one input is required, the I.C.$_-$ for the additional outputs cannot be arbitrarily chosen if these responses are to correspond to the same initial state of the system. The method of obtaining the I.C.$_-$ depends on how the initial state of the system is specified.

A system composed of elements which are equivalent to R-L-C elements can be handled most conveniently by state variables since the initial state is often given in terms of the initial values of the state variables. The next section of this chapter considers the state variable approach.

5.5 State Variables

5.5.1 Introduction

The state variable approach to system analysis is most advantageous for complex systems. Even for very simple systems where the classical time domain and transform methods are more direct than state variable analysis, the state space approach offers an intuitive feel for a problem. As the complexity of the system increases, computation by computer becomes the only feasible way to solve such analysis problems. Not only does the state variable approach lend itself to complex linear systems, but it also provides a conceptual technique for handling non-linear systems.

5.5.2 Normal Form

If a system is linear, the set of system equations can be expressed in the "Linear Normal Form"[17]. This matrix representation is

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \cdot \\ \cdot \\ \cdot \\ \dot{q}_K \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1K} \\ a_{21} & a_{22} \cdots & a_{2K} \\ a_{31} & a_{32} \cdots & a_{3K} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_{K1} & a_{K2} & a_{KK} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ \cdot \\ \cdot \\ \cdot \\ q_K \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \cdots & b_{1N} \\ b_{21} & b_{22} \cdots & b_{2N} \\ b_{31} & b_{32} \cdots & b_{3N} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ b_{K1} & b_{K2} & b_{KN} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \cdot \\ \cdot \\ \cdot \\ x_N \end{bmatrix} \quad (43)$$

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(17) Reference 5, page 37.

$$\begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_M \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1K} \\ c_{21} & c_{22} & \cdots & c_{2K} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ c_{M1} & c_{M2} & \cdots & c_{MK} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \cdot \\ \cdot \\ \cdot \\ q_K \end{bmatrix} + \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1N} \\ d_{21} & d_{22} & \cdots & d_{2N} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ d_{M1} & d_{M2} & \cdots & d_{MN} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_N \end{bmatrix} \quad (44)$$

where y_i $(i = 1,2,\ldots,M)$ are output variables

x_i $(i = 1,2,\ldots,N)$ are input variables

q_i $(i - 1,2,\ldots,K)$ are state variables

and a_{ij}, b_{ij}, c_{ij}, d_{ij} are in general functions of time

There are various procedures for obtaining a state variable representation of the above form. If the system is also time-invariant and a single-input, single-output system characterized by the differential equation

$$(D^k + \alpha_1 D^{k-1} + \ldots + \alpha_k)\, y(t) = (\beta_o D^k + \beta_1 D^{k-1} + \ldots + \beta_k) x(t) \quad (45)$$

a standard algorithm is available for generating the $\underline{A}$, $\underline{B}$, $\underline{C}$, and $\underline{D}$ matrices. These matrices are[18]

$$\underline{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_k & -\alpha_{k-1} & -\alpha_{k-2} & & -\alpha_1 \end{bmatrix} \qquad \underline{B} = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_k \end{bmatrix} \quad (46)$$

(18) Reference 5, pages 40-41.

$$\underline{C} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \end{bmatrix} \qquad \underline{D} = \begin{bmatrix} d \end{bmatrix} \tag{46}$$

where

$$\begin{aligned}
d \qquad\qquad\qquad\qquad\qquad\qquad\qquad &= \beta_o \\
\alpha_1 d + b_1 \qquad\qquad\qquad\qquad\qquad &= \beta_1 \\
\alpha_2 d + \alpha_1 b_1 + b_2 \qquad\qquad\qquad &= \beta_2 \\
&\;\;\vdots \\
\alpha_{k-1} d + \alpha_{k-2} b_1 + \alpha_{k-3} b_2 + \dots\dots + b_{k-1} &= \beta_{k-1} \\
\alpha_k d + \alpha_{k-1} b_1 + \alpha_{k-2} b_2 + \dots + \alpha_1 b_{k-1} + b_k &= \beta_k
\end{aligned} \tag{47}$$

One of the major advantages of using state variables is that the linear normal form results in a set of first order equations which can be handled either through manipulation of the matrix equations, Equations (43) and (44), to obtain the fundamental matrix, or through iterative numerical computations in the time domain.

If the conditions of the system represented by Equation (45) are given in terms of $y(t)$, $y^{(1)}(t)$, ..., $y^{(k-1)}(t)$ at $t=0_-$, the I.C.$_+$ in terms of the output can be obtained using the algorithms previously developed and then the corresponding I.C.$_+$ of the state variables can be found using the algorithm of Equation (48).

$$q_1(0_+) = y(0_+) - \beta_o x(0_+)$$

$$q_2(0_+) = y^{(1)}(0_+) - \beta_o x^{(1)}(0_+) - b_1 x(0_+)$$

$$q_3(0_+) = y^{(2)}(0_+) - \beta_o x^{(2)}(0_+) - b_1 x^{(1)}(0_+) - b_2 x^{(2)}(0_+) \qquad (48)$$

$$\vdots$$

$$q_k(0_+) = y^{(k-1)}(0_+) - \beta_o x^{(k-1)}(0_+) - b_1 x^{(k-2)}(0_+) - \ldots - b_{k-1} x^{(k)}(0_+)$$

The above algorithms for generating the state variable matrices do not represent a unique state variable assignment for the system in that there are numerous possible assignments for a given system. The particular choice of state variables resulting from these algorithms can be simulated conveniently on an analog computer.

5.5.3 Application to Ideal R-L-C Circuits

To simplify analysis, the system being considered can often be represented as being composed of ideal R-L-C elements. The state concept provides a simple procedure for analyzing the circuit since each voltage across a capacitor and each current through an inductor can be considered as a state variable. One usually thinks of the initial state of such a system in terms of initial charge or voltage on capacitors and initial currents in inductors. With practice, one can obtain the state variable equations by inspection of the circuit. The $I.C._-$ are taken as the initial values of the state variables. If the system is simple, it will be apparent what the $I.C._+$ will be. Otherwise one can obtain the differential equations corresponding to the circuit by manipulation of Equations (43) and (44). From Equation (44), for the single output case,

$$y = c_1 q_1 + c_2 q_2 + \ldots + c_K q_K + d_1 x_1 + d_2 x_2 + \ldots + d_N x_N \tag{49}$$

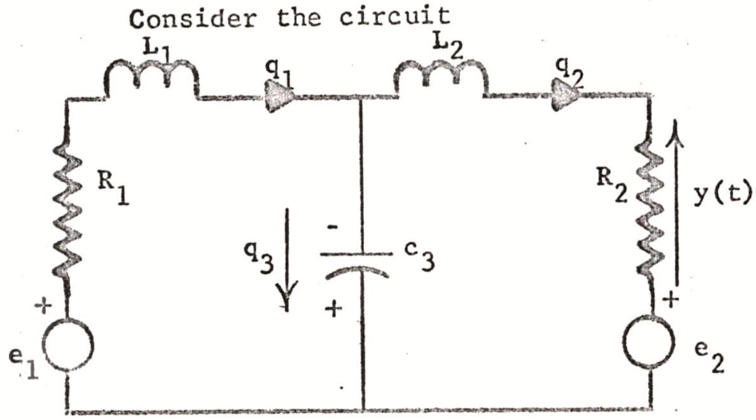
Differentiating Equation (49) we get

$$\dot{y} = c_1 \dot{q}_1 + c_2 \dot{q}_2 + \ldots + c_K \dot{q}_K + d_1 \dot{x}_1 + d_2 \dot{x}_2 + \ldots + d_N \dot{x}_N$$

One then substitutes the expression for the derivatives of the state variables using Equation (43). The differentiation and substitution is repeated until k simultaneous equations are obtained. One can then solve for $y^{(k)}(t)$ in terms of y(t) and its k-1 derivatives by manipulating these equations. The I.C.$_-$ of the output are found by using these same equations for the input functions zero. After the I.C.$_+$ are computed using the techniques discussed earlier, the reverse process is used to obtain the I.C.$_+$ for the state variables. The following example illustrates this procedure.

5.5.4 Example

Consider the circuit



$L_1 = L_2 = 1$ Henry

$C_3 = .5$ Farad

$R_1 = 1$ ohm

$R_2 = 4$ ohm

$e_1(t) = \delta(t)$

$e_2(t) = U(t)$

Figure 4

State Variable Assignment for a Simple R-L-C Circuit which is initially at rest. The state variables have been chosen as currents through L_1 and L_2 and the voltage across C_3, as indicated. The output is the voltage across R_2. Then from the circuit

$$q_1 - q_2 + \frac{s}{2} q_3 = 0$$

$$E_1 + q_3 - q_1(s+1) = 0$$

$$E_2 + q_3 + (s+4)q_2 = 0$$

$$y = 4q_2$$

The state variable equations are

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -4 & -1 \\ -2 & 2 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}$$

$$\begin{bmatrix} y(t) \end{bmatrix} = \begin{bmatrix} 0 & 4 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}$$

$$y = 4q_2$$

$$\dot{y} = 4\left[- 4q_2 - q_3 - E_2\right] = - 4y - 4q_3 - 4E_2$$

$$\ddot{y} = - 4\dot{y} - 4\left[- 2q_1 + 2q_2\right] - 4\dot{E}_2 = - 4\dot{y} + 8q_1 - 8q_2 - 4\dot{E}_2$$

$$\dddot{y} = - 4\ddot{y} + 8\left[- q_1 + q_3 + E_1 - \dot{q}_2\right] - 4\ddot{E}_2$$

$$= - 4\ddot{y} + 8\left[- q_2 + \tfrac{1}{2}\dot{q}_3 + q_3 + E_1 - \dot{q}_2\right] - 4\ddot{E}_2$$

$$= -4\ddot{y} - 4\ddot{E}_2 + 8\left[-q_2 - \dot{q}_2 + E_1 - \dot{q}_2 - 4q_2 - E_2 + \tfrac{1}{2}(-\dot{E}_2 - \ddot{q}_2 - 4\dot{q}_2)\right]$$

$$\dddot{y} = -4\ddot{y} - 4\ddot{E}_2 + 8\left[-5q_2 - 4\dot{q}_2 - \tfrac{1}{2}\ddot{q}_2 + E_1 - E_2 - \tfrac{1}{2}\dot{E}_2\right]$$

$$= -5\ddot{y} - 8\dot{y} - 10y - 8E_1 - 4\ddot{E}_2 - 4\dot{E}_2 - 8E_2$$

Using the general algorithm for modifying the initial conditions for this differential equation and the assumption that the system is initially ($t=0_-$) at rest, one finds

$$y(0_+) = 0$$

$$\dot{y}(0_+) = -4$$

$$\ddot{y}(0_+) = 24$$

Using the same equations

$$4q_2(0_+) = y(0_+) = 0$$

$$q_2(0_+) = 0$$

$$\dot{y}(0_+) + 4y(0_+) = -4q_3(0_+) - 4E_2(0_+)$$

$$4q_3(0_+) = 4 - (4)(1) = 0$$

$$q_3(0_+) = 0$$

$$\ddot{y}(0_+) = -4\dot{y}(0_+) + 8q_1(0_+) - 8q_2(0_+) - 4\dot{E}_2(0_+)$$

$$24 = (-4)(-4) + 8q_1(0_+) - 8(0) - 4(0)$$

$$q_1(0_+) = 1$$

5.5.5 Direct Modification of State Variable Initial Values

The state variables just as the I.C.$_-$ can often be determined directly by inspection of the system, i.e., without first obtaining and solving the differential equations of the system, and then generating the state equations. Hence, it would be more convenient to be able to obtain the I.C.$_+$ of the state variables from the corresponding I.C.$_-$ of the state variables without going back to the differential equation of the output. In the previous example, recall the considerable work to obtain the differential equation from the state equation. For a complex system such as a multiloop circuit, this procedure could be an exhaustive task. To make the state variable approach much more convenient as well as useful, we now proceed to a technique of determining the I.C.$_+$ directly.

If a system is linear and time invariant, the matrix Equation (43) can be Laplace transformed in a manner similar to that of a scalar equation. Thus we have

$$s\underline{I}\underline{Q}(s) - \underline{q}(0_-) = \underline{A}\underline{Q}(s) + \underline{B}\underline{X}(s) \qquad (50)$$

$$\left[s\underline{I} - \underline{A}\right]\underline{Q}(s) = \underline{B}\underline{X}(s) + \underline{q}(0_-)$$

where $\underline{I}$ is the identity matrix. If the state variables are linearly independent, it is clear that

$$\det\left[s\underline{I} - \underline{A}\right] \neq 0 ,$$

and therefore the matrix

$$\left[s\underline{I} - \underline{A}\right]^{-1}$$

exists and corresponds to the Laplace transform of the fundamental matrix, $e^{\underline{A}t}$. With the assumption of linear independence, we have

$$\underline{Q}(s) = \left[s\underline{I} - \underline{A}\right]^{-1} \underline{BX}(s) + \left[s\underline{I} - \underline{A}\right]^{-1} \underline{q}(0_-) \tag{51}$$

Applying the modified initial-value theorem to Equation (51), and recalling that for zero input

$$q(0_+) = q(0_-) \; ,$$

we get

$$\underline{q}(0_+) = \underline{q}(0_-) + \lim_{s \to \infty}{}^{*} \left[s\underline{I}(s\underline{I} - \underline{A})^{-1} \underline{BX}(s) \right] \tag{52}$$

where the limit is taken in the sense of Theorem 2. Comparing Equation (52) to Equation (22), with k=0, m=0, n=1, and $a_o = -1$, the two equations are very similar. In Equation (52), the limit is taken for each element of the matrix indicated by the brackets. It is important to note that once a convenient set of state variables is obtained for a system, along with the corresponding I.C.$_-$'s, the I.C.$_+$'s for the system can be found directly using Equation (52) even for a multiple-input, multiple-output system. The minimum number of state variables that are necessary to completely characterize an entire system will usually be obtained. If the state variables are not all linearly independent, the determinant

$$\det \left[s\underline{I} - \underline{A}\right] = 0$$

and it will be apparent from matrix algebra which of the state variables are redundant.

5.5.6 Example Using Direct Modification of State Initial Values

Consider the circuit of the previous example in section 5.5.4 .

We have

$$\left[s\underline{I} - \underline{A}\right] = \begin{bmatrix} s+1 & 0 & -1 \\ 0 & s+4 & 1 \\ 2 & -2 & s \end{bmatrix} , \quad \underline{B} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}$$

$$\underline{X} = \begin{bmatrix} 1 \\ \frac{1}{s} \end{bmatrix} \qquad q(0_-) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Then

$$\left[s\underline{I} - \underline{A}\right]^{-1} = \frac{\begin{bmatrix} (s^3+ 4s^2+ 2s) & (2s) & (s^2+ 4s) \\ (2s) & (s^3+ s^2+ 2s) & -s(s+1) \\ -2(s^2+ 4s) & 2s(s+1) & (s^3+ 5s^2+ 4s) \end{bmatrix}}{(s^3+ 5s^2+ 8s + 10)}$$

Substituting these matrices into Equation (52), we have

$$\begin{bmatrix} q_1(0_+) \\ q_2(0_+) \\ q_3(0_+) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \lim_{s \to \infty}{}^{*} \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} \left[s\underline{I} - \underline{A}\right]^{-1} \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{s} \end{bmatrix}$$

$$= \lim_{s \to \infty}{}^{*} \begin{bmatrix} \left(\dfrac{s^3+ 4s^2+ 2s - 2}{s^3+ 5s^2+ 8s + 10}\right) \\ \left(\dfrac{2s - s^2- s - 2}{s^3+ 5s^2+ 8s + 10}\right) \\ \left(\dfrac{-2s^2- 8s - 2s - 2}{s^3+ 5s^2+ 8s + 10}\right) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

5.6 Piecewise Linear Inputs

The type of inputs so far considered have been singularity functions with the origin as the only singular point. One may wish to analyze a system with an input which is composed of several singularity functions not applied at the same time. Such an input can be expressed as

$$x(t) = U(t)x_o(t) + U(t-t_1)x_1(t-t_1) + \ldots + U(t-t_n)x_n(t-t_n)$$

where

$$0 < t_1 < t_2 < \ldots < t_n$$

A straightforward technique of handling such an input is to apply the superposition theorem. That is, calculate the effect of each singularity and any derivatives associated with the I.C. on the output, and take their sum. If numerical integration is being used, the effect of each singularity is simply added to the response after the point of its occurrance is reached. After this point, the input function associated with the singularity is included in the total input function. A similar procedure is followed if the solution is obtained through classical time domain techniques. The conditions of the system are found at a point just before a given singularity occurs. These conditions are then modified through the techniques of the modified initial-value theorem, Theorem 2, to account for the effects of the singularity.

5.7 Systems with Delays

In general, delay paths in a system are difficult to handle by conventional techniques. The initial values of a function containing a delay element presents no problem since the path or loop in the system containing the delay can be considered as an open circuit. If the path containing the delay has no feedback paths associated with it, the delay element can be, in effect, pushed back through the system until its input is simply treated as part of the input of the system. Only in this case, can a system with a delay element be handled easily when finding the transient response since in this case the delay is incorporated within the input function.

5.7.1 Example of a System Having Delay

Consider the block diagram of a system with delay

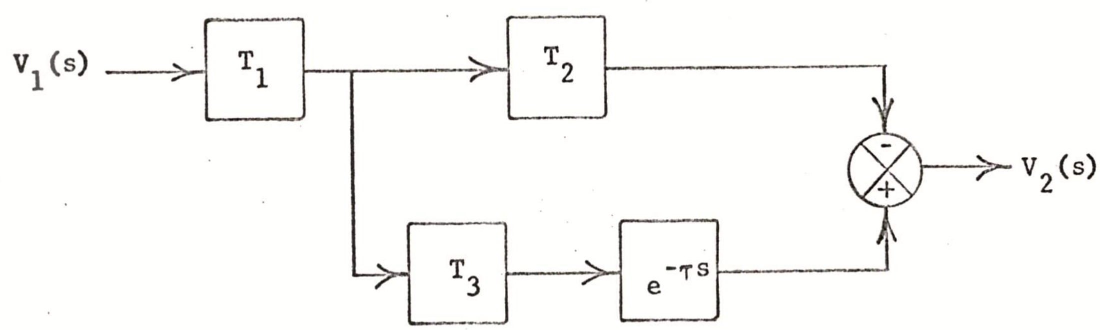


Figure 5

Block Diagram of a System with Delay

We assume that the system is linear and time-invariant and that the transfer functions, T_1, T_2, and T_3 have no delays. To push the element back to the input, block diagram transformation is accomplished using standard techniques[19]. Splitting the node which follows T_1, and then pushing the delay backwards, the block diagram becomes

(19) Reference 10, pages 262-267.

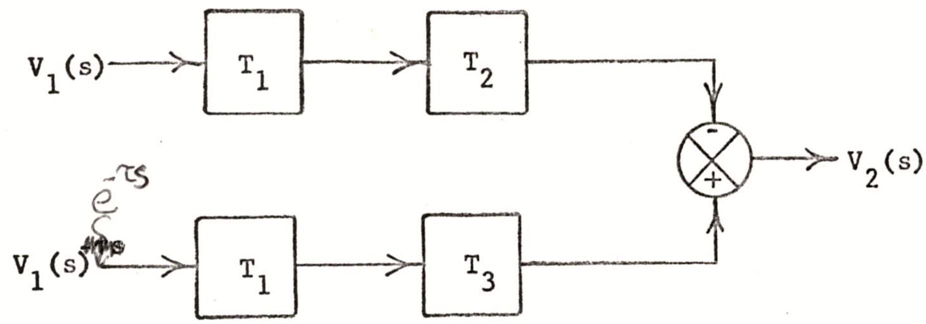


Figure 6

Transformed Block Diagram of Figure 5

It is clear that as long as the delay element is not within a feedback loop, the delay can be pushed back to the input. The delayed input is then considered as a new input function.

5.8 Time-Varying Systems

Time-varying systems are characterized by the differential equations having coefficients which are functions of time. Except for the simplest types of time-varying systems, solutions must be determined by approximate techniques such as series expansions and numerical integration. The response to an input function cannot be uniquely determined from just a knowledge of the system elements, input, and stored energy initially in the system as was done for a fixed system, because now the response depends also on the exact time of application of the input. This of course can be implicitly specified through additional initial conditions. In the usual treatment of time-varying systems given in textbooks, the initial conditions are considered simply as arbitrary constants.

If the coefficients of the differential equation of a system are continuous functions of time, it is clear that they can be considered constant over the infinitesmal interval during which the input is singular. They can be evaluated at this point of singularity since they are not functions of the output. A simplification that can be used if the coefficients change slowly is to consider them as piecewise-constant functions over small but finite intervals of time[20]. As the transition from one of these intervals to the next takes place, the coefficients go through finite discontinuous jumps.

If the time-varying coefficients are polynomials of time, Laplace transforms can be sometimes successfully used to obtain the transient response. Using the property

$$L\left\{ t^k f(t) \right\} = (-1)^k \frac{d^k F(s)}{ds^k}$$

where

$$F(s) = L\left\{ f(t) \right\}$$

the time domain differential equation is transformed to a k^{th} order differential equation in terms of F(s), where k depends on the highest power of t in the time domain differential equation. It is clear that additional conditions may be needed for the solution of this new differential equation and that this equation may be more difficult to solve than the original equation.

A time-varying system in a practical engineering application

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(20) Reference 4, page 469.

will usually be more complex than can be efficiently handled by the analytic techniques of analysis. Numerical analysis can often give a general indication of the system's performance, but analogue simulation is the most practical approach to this and more complex problems.

6. CONCLUSIONS

The problem of relating the initial conditions just before and just after time zero for linear systems has been investigated. It is often the case that the $I.C._-$ are different from the $I.C._+$ when the input to the system is singular at the origin. The modification necessary to obtain the positive-limit values has been accomplished through consideration of the initial-value property of the Laplace transform, which properly takes into account singularities at time zero.

Algorithms have been developed for treating linear, fixed systems from both the differential equation and the state variable standpoint, as an extension of the way in which the Laplace transform accounts for singularities in the output and its derivatives. Using these algorithms, one can directly and conveniently obtain the $I.C._+$ for the linear, fixed system from the known $I.C._-$. This technique is especially useful for the case of a piecewise-linear input which has several singularity points.

Since analysis of linear systems is often accomplished with digital computers, the algorithms have been programmed in Fortran IV. Examples are used throughout the thesis to clarify the results of various sections. Special consideration is given to R-L-C circuits since their initial conditions can be directly interpreted in terms of currents and voltages of circuit elements. A final generalization of this approach considers the modification of initial conditions for more complex linear systems of engineering interest such as those with delay and multiple inputs.

APPENDIX 1

EXPANSION ALGORITHM

A1.1 Derivation of Algorithm

Given the rational function

$$F(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_o}{s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_o}, \quad m \geq n \tag{53}$$

Dividing the numerator by the denominator yields

$$F(s) = b_m s^{m-n} + \left\{ (b_{m-1} - b_m a_{n-1}) s^{m-1} + (b_{m-2} - b_m a_{n-2}) s^{m-2} + \ldots + (b_{m-n} - b_m a_o) s^{m-n} + b_{m-n-1} s^{m-n-1} + \ldots + b_1 s + b_o \right\} \Big/ \left\{ s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_o \right\}$$

Letting

$$c_o = b_m$$

$$b'_{m-1} = b_{m-1} - c_o a_{n-1}$$

$$b'_{m-2} = b_{m-2} - c_o a_{n-2}$$

$$\vdots$$

$$b'_{m-n} = b_{m-n} - c_o a_o$$

$$b'_{m-n-1} = b_{m-n-1}$$

$$b'_{m-n-2} = b_{m-n-2}$$

$$\vdots$$

$$b'_o = b_o$$

We can write Equation (53) as

$$F(s) = c_o s^{m-n} + \frac{b'_{m-1}s^{m-1} + b_{m-2}s^{m-2} + \dots + b_1 s + b_o}{s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_o} \quad (54)$$

Repeating the division process, we obtain

$$F(s) = c_o s^{m-n} + b'_{m-1}s^{m-n-1} + \left\{(b'_{m-2} - b'_{m-1}a_{n-1})s^{m-2} + (b'_{m-3} - b'_{m-1}a_{n-2})s^{m-3} + \dots + (b'_{m-n-1} - b'_{m-1}a_o)s^{m-n-1} + b_{m-n-2}s^{m-n-2} + \dots + b_1 s + b_o\right\} \Big/ \left\{s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_o\right\}$$

For convenience, let

$$c_1 = b'_{m-1} = b_{m-1} - c_o a_{n-1} ,$$

and redefine

$$b'_{m-2} = b'_{m-2} - b'_{m-1}a_{n-1} = b_{m-2} - c_o a_{n-2} - c_1 a_{n-1}$$

$$b'_{m-3} = b_{m-3} - c_o a_{n-3} - c_1 a_{n-2}$$

$$b'_{m-4} = b_{m-4} - c_o a_{n-4} - c_1 a_{n-3}$$

$$\vdots$$

$$b'_{m-n} = b_{m-n} - c_o a_o - c_1 a_1$$

$$b'_{m-n-1} = b_{m-n-1} - c_1 a_o$$

$$b'_{m-n-2} = b_{m-n-2}$$

$$\vdots$$

$$b'_1 = b_1$$

$$b'_o = b_o$$

Then Equation (54) becomes

$$F(s) = c_o s^{m-n} + c_1 s^{m-n-1} + \frac{b'_{m-2} s^{m-2} + b'_{m-3} s^{m-3} + \ldots + b_o}{s^n + a_{n-1} s^{n-1} + \ldots + a_o} \tag{55}$$

To express the result of dividing the numerator by the denominator in Equation (55), it is again convenient to let

$$c_2 = b'_{m-2} = b_{m-2} - c_o a_{n-2} - c_1 a_{n-1}$$

and redefine

$$\begin{aligned}
b'_{m-3} &= b_{m-3} - c_o a_{n-3} - c_1 a_{n-2} - c_2 a_{n-1} \\
b'_{m-4} &= b_{m-4} - c_o a_{n-4} - c_1 a_{n-3} - c_2 a_{n-2} \\
b'_{m-5} &= b_{m-5} - c_o a_{n-5} - c_1 a_{n-4} - c_2 a_{n-3} \\
&\vdots \\
b'_{m-n} &= b_{m-n} - c_o a_o - c_1 a_1 - c_2 a_2 \\
b'_{m-n-1} &= b_{m-n-1} - c_1 a_o - c_2 a_1 \\
b'_{m-n-2} &= b_{m-n-2} - c_2 a_o \\
b'_{m-n-3} &= b_{m-n-3} \\
&\vdots \\
b'_1 &= b_1 \\
b'_o &= b_o
\end{aligned}$$

Continuing this process, the result of the next cycle is expressed in terms of

$$c_3 = b_{m-3} - c_o a_{n-3} - c_1 a_{n-2} - c_2 a_{n-1}$$

and the redefined coefficients

$$b'_{m-4} = b_{m-4} - c_o a_{n-4} - c_1 a_{n-3} - c_2 a_{n-2} - c_3 a_{n-1}$$

$$b'_{m-5} = b_{m-5} - c_o a_{n-5} - c_1 a_{n-4} - c_2 a_{n-3} - c_3 a_{n-2}$$

$$\vdots$$

$$b'_{m-n} = b_{m-n} - c_o a_o - c_1 a_1 - c_2 a_2 - c_3 a_3$$

$$b'_{m-n-1} = b_{m-n-1} - c_1 a_o - c_2 a_1 - c_3 a_2$$

$$b'_{m-n-2} = b_{m-n-2} - c_2 a_o - c_3 a_1$$

$$b'_{m-n-3} = b_{m-n-3} - c_3 a_o$$

$$b'_{m-n-4} = b_{m-n-4}$$

$$\vdots$$

$$b'_1 = b_1$$

$$b'_o = b_o$$

By now the pattern is established. For $0 < k \leq m-n$, we obtain

$$c_k = b_{m-k} - c_o a_{n-k} - c_1 a_{n-k+1} - c_2 a_{n-k+2} - \cdots - c_j a_{n-k+j} - \cdots$$

$$- c_{k-2} a_{n-2} - c_{k-1} a_{n-1} = b_{m-k} - \sum_{j=0}^{k-1} c_j a_{n+j-k} \; ; \quad k=1,2,\ldots,m-n$$

$a_r = 0$ for all $r < 0$

Combining the additional restriction on a_r yields

$$c_k = b_{m-k} - \sum_{j=k-n}^{k-1} c_j a_{n+j-k} \quad ; \quad k=1,2,\ldots,m-n$$

Letting i=j-k

$$c_k = b_{m-k} - \sum_{i=-n}^{-1} c_{k+i} a_{n+i} = b_{m-k} - \sum_{i=1}^{n} c_{k-i} a_{n-i}$$

Since $c_j = 0$ for all $j < 0$, we restrict the upper limit of the summation and get

$$c_o = b_m$$

$$c_k = b_{m-k} - \sum_{i=1}^{k} c_{k-i} a_{n-i} \; ; \quad k=1,2,\ldots,m-n \qquad (56)$$

$$a_r = 0 \text{ for all } r < 0$$

To determine the coefficients of the reduced fraction, we find the coefficients b_j' for the last cycle. Observing the pattern developed in the first three cycles, we obtain

$$b_{n-1}' = b_{n-1} - c_o a_{2n-m-1} - c_1 a_{2n-m} - \cdots - c_k a_{2n-m-1+k} - \cdots - c_{m-n} a_{n-1} =$$

$$b_{n-1} - \sum_{k=0}^{m-n} c_k a_{2n-m-1+k} \quad ; \quad a_r = 0 \text{ for all } r < 0$$

$$b_{n-2}' = b_{n-2} - c_o a_{2n-m-2} - c_1 a_{2n-m-1} - \cdots - c_k a_{2n-m-2+k} - \cdots - c_{m-n} a_{n-2} =$$

$$b_{n-2} - \sum_{k=0}^{m-n} c_k a_{2n-m-2+k} \quad ; \quad a_r = 0 \text{ for all } r < 0$$

Or, in general

$$b'_{n-j} = b_{n-j} - \sum_{k=0}^{m-n} c_k a_{2n-m-j+k} \; ; \qquad a_r = 0 \text{ for all } r < 0$$

To simplify the notation, let $n-j = i$

$$b'_i = b_i - \sum_{k=0}^{m-n} c_k a_{n-m+k+i} \qquad ; \qquad a_r = 0 \text{ for all } r < 0$$

and also denote $d_i = b'_i$.

We have converted Equation (53) to

$$F(s) = c_o s^{m-n} + c_1 s^{m-n-1} + \ldots + c_k s^{m-n-k} + \ldots + c_{m-n} +$$

$$\frac{d_{n-1} s^{n-1} + d_{n-2} s^{n-2} + \ldots + d_1 s + d_o}{s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_o} \tag{57}$$

where

$$c_o = b_m$$

$$c_k = b_{m-k} - \sum_{i=1}^{k} c_{k-i} a_{n-i} \; ; \qquad k=1,2,3,\ldots,m-n \qquad a_r = 0 \text{ for all } r < 0 \tag{58}$$

$$d_i = b_i - \sum_{k=0}^{m-n} c_k a_{n-m+k+i} \; ; \qquad i=0,1,\ldots,n-1 \qquad a_r = 0 \text{ for all } r < 0 \tag{59}$$

If $m < n$, Equation (53) is already in this form.

A1.2 Example

Given

$$F(s) = \frac{2s^5 + s^4 + 3s^2 + s + 1}{s^3 + s^2 - s + 1}$$

then the coefficients are

$b_o = 1$	$b_3 = 0$	$a_o = 1$	$m = 5$
$b_1 = 1$	$b_4 = 1$	$a_1 = -1$	$n = 3$
$b_2 = 3$	$b_5 = 2$	$a_2 = 1$	$m-n = 2$

Using Equation (58)

$$c_o = 2$$

$$c_k = b_{5-k} - \sum_{i=1}^{k} c_{k-i}a_{3-i} \quad ; \quad k=1,2$$

$$a_r = 0 \text{ for all } r < 0$$

$$c_1 = 1 - (2)(1) = -1$$

$$c_2 = 0 - (-1)(1) - (2)(-1) = 3$$

Using Equation (59)

$$d_i = b_i - \sum_{k=0}^{2} c_k a_{k+i-2} \quad ; \quad i=0,1,2$$

$$a_r = 0 \text{ for all } r < 0$$

$$d_o = 1 - (3)(1) = -2$$

$$d_1 = 1 - (-1)(1) - (3)(-1) = 5$$

$$d_2 = 3 - (2)(1) - (-1)(-1) - (3)(1) = -3$$

Therefore we can write the given function as

$$F(s) = 2s^2 - s + 3 + \frac{-3s^2 + 5s - 2}{s^3 + s^2 - s + 1}$$

This result is easily checked by synthetic division.

APPENDIX 2

DERIVATION OF INITIAL CONDITIONS ALGORITHM

A2.1 General Equations

Given

$$F(s) = \frac{B_p s^p + B_{p-1} s^{p-1} + \ldots + B_o}{s^q + A_{q-1} s^{q-1} + \ldots + A_o} = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_o}{s^n + a_{n-1} s^{n-1} + \ldots + a_o} \cdot X(s) \qquad (60)$$

and from Chapter 4, Equation (30), $y^{(k)}(0_+) = y^{(k)}(0_-) + c_{p+k+1-q}$

where $c_o = B_p$

$$c_j = B_{p-j} - \sum_{i=1}^{j} c_{j-i} A_{q-i} \; ; \qquad j = 1,2,\ldots,p+n-q \qquad (61)$$

$$c_r, B_r, A_r = 0 \text{ for all } r < 0$$

and Equation (31) regarding the impulses of $f^{(k)}(t)$ at the origin of

order	and	magnitude
$p+k-q$		c_o
$p+k-q-1$		c_1
.		.
.		.
.		.
$p+k-j-q$		c_j
.		.
.		.
.		.
1		$c_{p+k-q-1}$
0		c_{p+k-q}

(31)

find expressions similar to Equations (30) and (31) in terms of the input constants and the constants.

A2.2 Impulse Function (L^{th} Order)

$$x(t) = \alpha\, \delta^{(L)}(t)$$

then

$$X(s) = \alpha\, s^{L} = \alpha_1 s^{L} \qquad \alpha_1 = \alpha$$

and Equation (60) becomes after multiplication

$$F(s) = \frac{\alpha_1 b_m s^{m+L} + \alpha_1 b_{m-1} s^{m+L-1} + \ldots + \alpha_1 b_o s^{L}}{s^n + a_{n-1} s^{n-1} + \ldots + a_o}$$

Comparing to the left hand side of Equation (60)

$$p = m+L$$

$$q = n$$

$$A_j = a_j$$

$$\begin{aligned}
B_o &= 0 \\
&\vdots \\
B_{L-1} &= 0 \\
B_L &= \alpha_1 b_o \\
B_{L+1} &= \alpha_1 b_1 \\
&\vdots \\
B_r &= \alpha_1 b_{r-L} \\
&\vdots \\
B_{m+L} &= \alpha_1 b_m
\end{aligned}$$

from Equation (61)

$$c_o = \alpha_1 b_m$$

$$c_j = \alpha b_{m-j} - \sum_{i=1}^{j} c_{j-i} a_{n-i} \; ; \qquad j=1,2,\ldots,m+L$$

$$a_r, b_r = 0 \text{ for all } r < 0$$

$$y^{(k)}(0_+) = y^{(k)}(0_-) + c_{m+L+1+k-n} \qquad c_r = 0 \text{ for all } r < 0$$

from Equation (31), $y^{(k)}(t)$ has impulses at the origin of

order	and	magnitude
m+L+k-n		c_o
.		.
.		.
.		.
1		$c_{m+L+k-n-1}$
0		$c_{m+L+k-n}$

A2.3 Power of t: $x(t) = \alpha t^L$

$$X(s) = \frac{\alpha(L!)}{s^{L+1}} = \frac{\alpha_1}{s^{L+1}} \qquad \alpha_1 = (\alpha)(L!)$$

Equation (60) becomes

$$F(s) = \frac{\alpha_1 b_m s^m + \alpha_1 b_{m-1} s^{m-1} + \ldots + \alpha_1 b_o}{s^{n+L+1} + a_{n-1} s^{n+1} + a_{n-2} s^{n+L-1} + \ldots + a_o s^{L+1}}$$

Comparing to the left hand side of Equation (60)

$$B_j = b_j \alpha_1$$

$$p = m$$

$$q = n+L+1$$

$$A_o = 0$$
$$\vdots$$
$$A_L = 0$$
$$A_{L+1} = a_o$$
$$A_{L+2} = a_1$$
$$\vdots$$
$$A_j = a_{j-L-1}$$
$$\vdots$$
$$A_{n+L} = a_{n-1}$$

From Equation (61)

$$c_o = \alpha_1 b_m$$

$$c_j = \alpha_1 b_{m-j} - \sum_{i=1}^{j} c_{j-i} a_{n-i} \quad ; \quad j = 1,2,\ldots,m-L-1$$
$$a_r = 0 \text{ for all } r < 0$$

$$y^{(k)}(0_+) = y^{(k)}(0_-) + c_{m+k-n-L} \qquad c_r = 0 \text{ for all } r < 0$$

$y^{(k)}(t)$ has impulses at the origin of

order	and	magnitude
$m+k-n-L-1$		c_o
$\vdots$		$\vdots$
1		$c_{m+k-n-L-2}$
0		$c_{m+k-n-L-1}$

A2.4 Exponential Function: $x(t) = \alpha e^{\beta t}$

$$X(s) = \frac{\alpha}{s-\beta} = \frac{\alpha_1}{s-\beta} \qquad \alpha_1 = \alpha$$

$$F(s) = \frac{\alpha_1 b_m s^m + \alpha_1 b_{m-1} s^{m-1} + \dots + \alpha_1 b_o}{s^{n+1} + (a_{n-1} - \beta) s^n + (a_{n-2} - \beta a_{n-1}) s^{n-1} + \dots + (a_o - \beta a_1) s - \beta a_o}$$

$$B_j = \alpha_1 b_j$$

$$p = m$$

$$q = n+1$$

$$A_n = (a_{n-1} - \beta)$$

$$A_{n-1} = (a_{n-2} - \beta a_{n-1})$$

$$\vdots$$

$$A_j = a_{j-1} - \beta a_j$$

$$\vdots$$

$$A_1 = a_o - \beta a_1$$

$$A_o = - \beta a_o$$

From Equation (61)

$$c_o = \alpha_1 b_m$$

$$c_j = \alpha_1 b_{m-j} - \sum_{i=1}^{j} c_{j-i} \left[a_{n-i} - \beta a_{n-i+1} \right] \; ; \; j=1,2,\dots,m-1$$

$$a_r = 0 \text{ for all } r < 0$$

$$a_n = 1$$

$$y^{(k)}(0_+) = y^{(k)}(0_-) + c_{m+k-n} \qquad c_r = 0 \text{ for all } r < 0$$

$y^{(k)}(t)$ has impulses at the origin of

order	and	magnitude
$m+k-n-1$		c_o
.		.
.		.
.		.
1		$c_{m+k-n-2}$
0		$c_{m+k-n-1}$

A2.5 Sinusoid: $x(t) = \alpha \sin \beta t$

$$X(s) = \frac{\alpha\,\beta}{s^2 + \beta^2} = \frac{\alpha_1}{s^2 + \beta^2} \qquad \alpha_1 = \alpha\,\beta$$

$$F(s) = \alpha_1 b_m s^m + \alpha_1 b_{m-1} s^{m-1} + \ldots + \alpha_1 b_o \Big/ \Big(s^{n+2} + a_{n-1} s^{n+1} + (a_{n-2} + \beta^2) s^n$$

$$+ (a_{n-3} + \beta^2 a_{n-1}) s^{n-1} + (a_{n-4} + \beta^2 a_{n-2}) s^{n-2} + \ldots +$$

$$(a_{j-2} + \beta^2 a_j) s^j + \ldots + (a_o + \beta^2 a_2) s^2 + \beta^2 a_1 s + \beta^2 a_o\Big)$$

$B_j = \alpha_1 b_j$

$p = m$

$q = n+2$

$A_{n+1} = a_{n-1}$

$A_n = a_{n-2} + \beta^2$

$$A_{n-1} = a_{n-3} + \beta^2 a_{n-1}$$

$$\vdots$$

$$A_j = a_{j-2} + \beta^2 a_j$$

$$\vdots$$

$$A_2 = a_o + \beta^2 a_2$$

$$A_1 = \beta^2 a_1$$

$$A_o = \beta^2 a_o$$

From Equation (61)

$$c_o = \alpha_1 b_m$$

$$c_j = \alpha_1 b_{m-j} - \sum_{i=1}^{j} c_{j-i} \left[a_{n-i} + \beta^2 a_{n+2-i} \right] ; \quad j=1,2,\ldots,m-2$$

where $a_r = 0$ for all $r < 0$

$a_r = 0$ for all $r > 0$

$a_r = 1 \qquad r = n$

$$y^{(k)}(0_+) = y^{(k)}(0_-) + c_{m+k-n-1} ; \qquad c_r = 0 \text{ for all } r \quad 0$$

$y^{(k)}(t)$ has impulses at the origin of

order	and	magnitude
$m+k-n-2$		c_o
$\vdots$		$\vdots$
1		$c_{m+k-n-3}$
0		$c_{m+k-n-2}$

A2.6 Cosine Function: $\alpha \cos \beta t = x(t)$

$$X(s) = \frac{\alpha s}{s^2 + \beta^2} = \frac{\alpha_1 s}{s^2 + \beta^2} \qquad \alpha_1 = \alpha$$

$$F(s) = \alpha_1 b_m s^{m+1} + \alpha_1 b_{m-1} s^n + \ldots + \alpha_1 b_o s \Big/ \Big(s^{n+2} + a_{n-1} s^{n+1} +$$

$$(a_{n-2} + \beta^2) s^n + (a_{n-3} + \beta^2 a_{n-1}) s^{n-1} + (a_{n-4} + \beta^2 a_{n-2}) s^{n-2} +$$

$$\ldots + (a_{j-2} + \beta^2 a_j) s^j + \ldots + (a_o + \beta^2 a_2) s^2 + \beta^2 a_1 s + \beta^2 a_o \Big)$$

$p = m+1 \qquad\qquad q = n+2$

$$\begin{aligned}
B_o &= 0 & A_{n+1} &= a_{n-1} \\
B_1 &= \alpha_1 b_o & A_n &= a_{n-2} + \beta^2 \\
B_2 &= \alpha_1 b_1 & A_{n-1} &= a_{n-3} + \beta^2 a_{n-1} \\
&\vdots & &\vdots \\
B_j &= \alpha_1 b_{j+1} & A_j &= a_{j-2} + \beta^2 a_j \\
&\vdots & &\vdots \\
B_{m+1} &= \alpha_1 b_m & A_2 &= a_o + \beta^2 a_2 \\
& & A_1 &= \beta^2 a_1 \\
& & A_o &= \beta^2 a_o
\end{aligned}$$

From Equation (61)

$$c_o = \alpha_1 b_m$$

$$c_j = \alpha_1 b_{m-j} - \sum_{i=1}^{j} c_{j-i}\left[a_{n-i} + \beta^2 a_{n+2-i}\right]; \quad j=1,2,\ldots,m-1$$

$a_r = 0$ for all $r < 0$

$a_r = 0$ for all $r > 0$

$a_n = 1$

$$y^{(k)}(0_+) = y^{(k)}(0_-) + c_{m+k-n}$$

$c_r = 0$ for all $r < 0$

$y^{(k)}(0_+)$ has impulses at the origin of

order	and	magnitude
$m+k-n-1$		c_o
.		.
.		.
.		.
1		$c_{m+k-n-2}$
0		$c_{m+k-n-1}$

A2.7 Hyperbolic Sine Function: $x(t) = \alpha \sinh \beta t$

$$X(s) = \frac{\alpha\beta}{s^2 - \beta^2} = \frac{\alpha_1}{s^2 - \beta^2} \qquad \alpha_1 = \alpha\beta$$

This function is the same as the sine function except that in the Laplace transform of x(t), the sign of β^2 is reversed. From the sine function derivation, one obtains for the hyperbolic sine

$$c_o = \alpha_1 b_m$$

$$c_j = \alpha_1 b_{m-j} - \sum_{i=1}^{j} c_{j-i}\left[a_{n-i} - \beta^2 a_{n+2-i}\right]; \quad j=1,2,\ldots,m-2$$

where $a_r = 0$ for all $r < 0$

$a_r = 0$ for all $r > 0$

$a_r = 1 \quad r = n$

$$y^{(k)}(0_+) = y^{(k)}(0_-) + c_{m+k-n-1} \qquad c_r = 0 \text{ for all } r < 0$$

$y^{(k)}(t)$ has impulses at the origin of

order	and	magnitude
m+k-n-2		c_o
.		.
.		.
.		.
1		$c_{m+k-n-3}$
0		$c_{m+k-n-2}$

A2.8 Hyperbolic Cosine Function: $x(t) = \alpha \cosh \beta t$

$$X(s) = \frac{\alpha\, s}{s^2 - \beta^2} = \frac{\alpha_1 s}{s^2 - \beta^2} \qquad \alpha_1 = \alpha$$

This function is the same as the cosine function except that in the Laplace transform of x(t), the sign of β is reversed. From the cosine function derivation, one obtains for the hyperbolic cosine

$$c_o = \alpha_1 b_m$$

$$c_j = \alpha_1 b_{m-j} - \sum_{i=1}^{j} c_{j-i}\left[a_{n-i} - \beta^2 a_{n+2-i}\right]; \quad j=1,2,\ldots,m-1$$

$a_r = 0$ for all $r < 0$

$a_r = 0$ for all $r > n$

$a_n = 1$

$$y^{(k)}(0_+) = y^{(k)}(0_-) + c_{m+k-n} \qquad c_r = 0 \text{ for all } r < 0$$

$y^{(k)}(t)$ has impulses at the origin of

order	and	magnitude
$m+k-n-1$		c_o
.		.
.		.
.		.
1		$c_{m+k-n-2}$
0		$c_{m+k-n-1}$

A2.9 Time-Exponential: $x(t) = \alpha t\, e^{\beta t}$

$$X(s) = \frac{\alpha}{(s-\beta)^2} = \frac{\alpha}{s^2 - 2s\beta + \beta^2} = \frac{\alpha_1}{s^2 - 2s\beta + \beta^2}, \quad \alpha_1 = \alpha$$

$$F(s) = \left\{\alpha_1 b_m s^m + \alpha_1 b_{m-1} s^{m-1} + \dots + \alpha_1 b_o\right\} \Big/ \left\{s^{n+2} + a_{n-1}s^{n+1} + (a_{n-2} + \beta^2)s^n\right.$$

$$+ (a_{n-3} + \beta^2 a_{n-1})s^{n-1} + (a_{n-4} + \beta^2 a_{n-2})s^{n-2} + \dots + (a_{j-2} + \beta^2 a_j)s^j$$

$$+ \dots + (a_o + \beta^2 a_2)s^2 + \beta^2 a_1 s + \beta^2 a_o - 2s^{n+1}\beta - 2\beta a_{n-1}s^n$$

$$\left. - \dots - 2\beta a_{j-1}s^j - \dots - 2\beta a_o s\right\}$$

$$B_j = \alpha_1 b_j \qquad m = p \qquad q = n+2$$

$$A_{n+1} = a_{n-1} - 2\beta$$

$$A_n = a_{n-2} - 2\beta a_{n-1} + \beta^2$$

$$A_{n-1} = a_{n-3} - 2\beta a_{n-2} + \beta^2 a_{n-1}$$

$$A_{n-2} = a_{n-4} - 2\beta a_{n-3} + \beta^2 a_{n-2}$$

$$\vdots \qquad \vdots$$

$$A_j = a_{j-2} - 2\beta a_{j-1} + \beta^2 a_j$$

$$\vdots \qquad \vdots$$

$$A_2 = a_o - 2\beta a_1 + \beta^2 a_2$$

$$A_1 = - 2\beta a_o + \beta^2 a_1$$

$$A_o = + \beta^2 a_o$$

From Equation (61)

$$c_o = \alpha_1 b_m$$

$$c_j = \alpha_1 b_{m-j} - \sum_{i=1}^{j} c_{j-1} \left[a_{n-i} - 2\beta a_{n+1-i} + \beta^2 a_{n+2-i} \right]$$

$j=1,2,\ldots,m-2$

$a_r = 0$ for all $r < 0$

$a_r = 0$ for all $r > n$

$a_n = 1$

$$y^{(k)}(0_+) = y^{(k)}(0_-) + c_{m+k-n-1} \qquad c_r = 0 \text{ for all } r < 0$$

$y^{(k)}(t)$ has impulses at the origin of

order	and	magnitude
$m+k-n-2$		c_o
⋮	⋮	⋮

order and	magnitude
.	.
.	.
1	$c_{m+k-n-3}$
0	$c_{m+k-n-2}$

A2.10 Exponential-Sine: $x(t) = \alpha e^{\beta t} \sin \gamma t$

$$X(s) = \frac{\alpha\gamma}{(s-\beta)^2 + \gamma^2} = \frac{\alpha_1}{s^2 - 2s\beta + (\beta^2 + \gamma^2)} \qquad \alpha_1 = \alpha\gamma$$

Compare this function of s to the one corresponding to the time-exponential function, $\alpha t e^{\beta t}$. Where β^2 appears in the derivation for $\alpha t e^{\beta t}$, replace β^2 by $\beta^2 + \gamma^2$. Hence,

$$B_j = \alpha_1 b_j$$

$$A_j = a_{j-2} - 2\beta a_{j-1} + (\beta^2 + \gamma^2) a_j$$

From Equation (61)

$$c_o = \alpha_1 b_m$$

$$c_j = \alpha_1 b_{m-j} - \sum_{i=1}^{j} c_{j-i} \left[a_{n-i} - 2\beta a_{n+1-i} + (\beta^2 + \gamma^2) a_{n+2-i} \right]$$

$j=1,2,\ldots,m-2$

$a_r = 0$ for all $r < 0$

$a_r = 0$ for all $r > n$

$a_n = 1$

$$y^{(k)}(0_+) = y^{(k)}(0_-) + c_{m+k-n-1} \qquad c_r = 0 \text{ for all } r < 0$$

$y^{(k)}(t)$ has impulses at the origin of

order	and	magnitude
$m+k-n-2$		c_o
.		.
.		.
.		.
1		$c_{m+k-n-3}$
0		$c_{m+k-n-2}$

A2.11 Exponential-Cosine Function: $x(t) = \alpha e^{\beta t}\sin \gamma t$

$$X(s) = \frac{\alpha(s-\beta)}{(s-\beta)^2 + \gamma^2} = \frac{\alpha_1(s-\beta)}{(s-\beta^2) + \gamma^2} \qquad \alpha_1 = \alpha$$

Compare this function of s to that of the exponential-sine function. Since the denominators are the same for these two functions of s, the corresponding product functions, F(s), will also have the same denominators. Thus

$$p = m+1 \qquad q = n+2$$

$$A_j = a_{j-2} - 2\beta a_{j-1} + (\beta^2 + \gamma^2)a_j$$

The numerator of F(s) is

$$\alpha_1 b_m s^{m+1} + \alpha_1(b_{m-1} - \beta b_m)s^m + \alpha_1(b_{m-2} - \beta b_{m-1})s^{m-1} + \ldots +$$

$$\alpha_1(b_{j-1} - \beta b_j)s^j + \ldots + \alpha_1(b_o - \beta b_1)s - \alpha_1 \beta b_o$$

From Equation (61)

$$c_o = \alpha_1 b_m$$

$$c_j = \alpha_1(b_{m-j} - \beta b_{m+1-j}) - \sum_{i=1}^{j} c_{j-i}\left[a_{n-i} - 2\beta a_{n+1-i} + (\beta^2 + \gamma^2)a_{n+2-i}\right]$$

$j = 1, 2, \ldots, m-1$

where $a_r = 0$ for all $r < 0$

$a_r = 0$ for all $r > n$

$a_n = 1$

$y^{(k)}(0_+) = y^{(k)}(0_-) + c_{m+k-n}$ $\qquad c_r = 0$ for all $r < 0$

$y^{(k)}(t)$ has impulses at the origin of

order	and	magnitude
$m+k-n-1$		c_o
.		.
.		.
.		.
1		$c_{m+k-n-2}$
0		$c_{m+k-n-1}$

A2.12 Time-Sinusoid Function: $x(t) = \alpha t \sin \beta t$

$$X(s) = \frac{2\alpha\beta s}{(s^2 + \beta^2)^2} = \frac{\alpha_1 s}{(s^2 + \beta^2)^2} \qquad \alpha_1 = 2\,\alpha\,\beta$$

$$F(s) = \left\{\alpha_1 s^{m+1} b_m + \alpha_1 b_{m-1} s^m + \ldots + \alpha_1 b_o s\right\} \Big/ \left\{(s^{n+4} + a_{n-1} s^{n+3} + a_{n-2} s^{n+2} + \right.$$

$$a_{n-3} s^{n+1} + a_{n-4} s^n + \ldots + a_{j-4} s^j + \ldots + a_o s^4) + (2\beta^2 s^{n+2} +$$

$$2\beta^2 a_{n-1} s^{n+1} + 2\beta^2 a_{n-2} s^n + \ldots + 2\beta^2 a_{j-2} s^j + \ldots + 2\beta^2 a_o s^2) +$$

$$\left.(\beta^4 s^n + \beta^4 a_{n-1} s^{n-1} + \ldots + \beta^4 a_j s^j + \ldots + \beta^4 a_o)\right\}$$

$$B_j = \alpha_1 b_{j-1} \qquad p = m+1$$

$$q = n+4$$

$$A_{n+3} = a_{n-1}$$

$$A_{n+2} = a_{n-2} + 2\beta^2$$

$$A_{n+1} = a_{n-3} + 2\beta^2 a_{n-1}$$

$$A_n = a_{n-4} + 2\beta^2 a_{n-2} + \beta^4$$

$$A_{n-1} = a_{n-5} + 2\beta^2 a_{n-3} + \beta^4 a_{n-1}$$

$$A_{n-2} = a_{n-6} + 2\beta^2 a_{n-4} + \beta^4 a_{n-2}$$

$$\vdots$$

$$A_j = a_{j-4} + 2\beta^2 a_{j-2} + \beta^4 a_j$$

$$\vdots$$

$$A_4 = a_o + 2\beta^2 a_2 + \beta^4 a_4$$

$$A_3 = 2\beta^2 a_1 + \beta^4 a_3$$

$$A_2 = 2\beta^2 a_o + \beta^4 a_2$$

$$A_1 = + \beta^4 a_1$$

$$A_o = + \beta^4 a_o$$

From Equation (61)

$$c_o = \alpha_1 b_m$$

$$c_j = \alpha_1 b_{m-j} - \sum_{i=1}^{j} c_{j-i} \left[a_{n-i} + 2\beta^2 a_{n+2-i} + \beta^4 a_{n+4-i} \right]$$

$$j = 1, 2, \ldots, m-3$$

$$a_r = 0 \text{ for all } r < 0$$

$$y^{(k)}(0_+) = y^{(k)}(0_-) + c_{m+k-n-2} \qquad \begin{array}{l} a_r = 0 \text{ for all } r > n \\ a_n = 1 \\ c_r = 0 \text{ for all } r < 0 \end{array}$$

$y^{(k)}(t)$ has impulses at the origin of

order	and	magnitude
$m+k-n-3$		c_o
.		.
.		.
.		.
1		$c_{m+k-n-4}$
0		$c_{m+k-n-3}$

A2.13 Time-Cosine Function: $x(t) = \alpha t \cos \beta t$

$$X(s) = \frac{\alpha(s^2 - \beta^2)}{(s^2 + \beta^2)^2} = \frac{\alpha_1(s^2 - \beta^2)}{(s^2 + \beta^2)^2} \qquad \alpha_1 = \alpha$$

Since the numerator is the same as that of the function of s corresponding to the time-sine function, it follows that

$$A_j = a_{j-4} + 2\beta^2 a_{j-2} + \beta^4 a_j$$

The numerator of F(s) becomes

$$\alpha_1 b_m s^{m+2} + \alpha_1 b_{m-1} s^{m+1} + \alpha_1(b_{m-2} - \beta^2 b_m) s^m + \alpha_1(b_{m-3} - \beta^2 b_{m-1}) s^{m-1} +$$

$$\ldots + \alpha_1(b_{j-2} - \beta^2 b_j) s^j + \ldots + \alpha_1(b_o - \beta^2 b_2) s^2 - \alpha_1\beta^2 b_1 s - \alpha_1 \beta^2 b_o$$

Then $B_j = \alpha_1(b_{j-2} - \beta^2 b_j)$, if $b_r = 0$ for all $r < 0$

" " $r > n$

From Equation (61)

$$c_o = \alpha_1 (b_m)$$

$$c_j = \alpha_1 (b_{m-j} - \beta^2 b_{m+2-j}) - \sum_{i=1}^{j} c_{j-i} \left[a_{n-i} + 2\beta^2 a_{n+2-i} + \beta^4 a_{n+4-i} \right]$$

$j=1,2,\ldots,m-2$

$a_r, b_r = 0$ for all $r < 0$

" " $r > n$

$a_n = 1$

$$y^{(k)}(0_+) = y^{(k)}(0_-) + c_{m+k-n-1} \qquad c_r = 0 \text{ for all } r < 0$$

$y^{(k)}(t)$ has impulses at the origin of

order	and	magnitude
$m+k-n-2$		c_o
.		.
.		.
.		.
1		$c_{m+k-n-3}$
0		$c_{m+k-n-2}$

APPENDIX 3

COMPUTER PROGRAMS

A3.1 Introduction

This appendix contains a computer program with associated subroutines and function subprograms. The purpose of the program is to compute the modified initial conditions and the impulse information for a single input/single output, linear, fixed system. The data required is the system transfer function, $I.C._-$, and the input function. The input function may be expressed as a standard function in the time domain or as its Laplace transform provided that the transformed function is the quotient of two polynomials in s. In the latter case, the general algorithm is used for computation. In the former case, the direct algorithm is used.

Subroutines ALPH, NUM, XPRINT, and FRACTN are of secondary importance since their purpose is only to print the input function and quotients of two polynomials in a conventional form. The purpose of the other subroutines and function subprograms is to compute the constants

$$c(I),\ I = 1,2,\ldots,J_m$$

of the initial-conditions algorithms. It is expected that if one should desire to use these algorithms in a programmed form, a main program for the specific use would differ from the one that has been used here.

A3.2 Data Representation

For the main program used, the data used by the program must be in the form dictated by the FORMAT statements. The sequencing of the data cards is summarized below, and example data and output for several simple systems is provided at the end of this appendix.

Data	Comments
M,N	These two constants represent the order of the numerator and denominator, respectfully, for the transfer function.
(B(M+2-I),I=1,M1),(A(N+2-I),I=2,N1)	These constants represent the coefficients of the two polynomials of the transfer function, where M1=M+1 and N1=N+1.
YL(I),I=1,N	These constants are the I.C._; i.e., $(YL(K)=y^{(K-1)}(0_-))$.
INP	This constant has the value zero if the Laplace transform of the input function is read. Otherwise its value is the input function number corresponding to the number in Table 1.

The remaining data cards represent the input function. If INP $\neq$ zero, a single card is used which is

ALPHA, BETA, G, L	These constants correspond to the constants of Table 1, where $G=\gamma$.

If INP is zero, the input function is represented by the data cards

MX, NX	These two constants represent the order of the numerator and denominator, respectfully, for the Laplace transform of the input.
(BX(MX+2-I),I=1,MX1), (AX(NX+2-I),I=1,NX1)	These constants represent the coefficients of the two polynomials of the Laplace transform of the input, where MX1=MX+1 and NX1=NX+1.

Each entry represents data to be put on one card according to the FORMAT of the appropriate input statement. However, when arrays are read, it is permissible to continue the data of these arrays on additional cards.

A3.3 Fortran Programs

A print-out of the main program, subroutines, and function subprograms follow.

```
C                      INITIAL CONDITIONS ALGORITHMS
C
C       THE PURPOSE OF THIS PROGRAM IS TO APPLY EITHER THE GENERAL
C       ALGORITHM OR THE DIRECT ALGORITHM DEPENDING ON THE INPUT.  A
C       TRANSFER FUNCTION IS READ AND THEN THE INITIAL CONDITIONS OF THE
C       OUTPUT AT  TIME=0 IS READ. AFTER THIS, AN INPUT OR ITS LAPLACE
C       TRANSFORM IS READ. IF THE LAPLACE TRANSFORM OF THE INPUT IS READ,
C       THEN THE GENERAL ALGORITHM IS USED. OTHERWISE THE DIRECT ALGORITHM
C       CORRESPONDING TO THE INPUT FUNCTION IS USED.
C           THE DATA READ IN IS PRINTED IN COMPREHENSIVE FORM. THEN THE
C       MODIFIED INITIAL CONDITIONS AND RELEVANT IMPULSE INFORMATION IS
C       PRINTED OUT IN A COMPREHENSIVE FORM. THE EXACT FORM OF THE DATA,
C       ALGORITHMS, AND OUTPUT IS REPRESENTED IN THIS APPENDIX. A FLOW-
C       CHART OF THE PROGRAM AND EXAMPLE RESULTS IS ALSO PRESENTED HERE.
C
        DIMENSION A(100),B(100),C(100),AX(100),BX(100),YL(100),YR(100)
      1 PRINT10
        READ 30,M,N
        M1=M+1
        N1=N+1
        READ 31,(B(M+2-I),I=1,M1),(A(N+2-I),I=2,N1)
        A(N1)=1
        PRINT 11
        CALL FRACTN(M,B,N,A)
        READ 31, (YL(I),I=1,N)
        PRINT 12
        READ 32, INP
        IF(INP.GT.0) GO TO 3
        READ 30, MX,NX
        MX1=MX+1
        NX1=NX+1
        READ 31,(BX(MX+2-I),I=1,MX1),(AX(NX+2-I),I=1,NX1)
        PRINT 13
        CALL FRACTN(MX,BX,NX,AX)
        GO TO 4
      3 READ 33,ALPHA,BETA,G,L
        CALL XPRINT(INP,ALPHA, BETA,G,L)
        CALL DIRALG(ALPHA,L,BETA,G,INP,M,N,B,A,JM,C)
        GO TO 5
      4 CALL MULT(B,BX,M,MX)
        CALL MULT(A,AX,N,NX)
C       THE TWO CARDS, PRINT 14 AND CALL FRACTN, CAN BE USED TO PRINT OUT
C       THE LAPLACE TRANSFORM OF THE  OUTPUT FUNCTION. THESE CARDS HAVE
C       BEEN MADE INTO COMMENT CARDS SINCE THE OUTPUT WOULD NOT FIT ON A
C       STANDARD SIZE PAGE FOR THE THESIS. THESE CARDS CAN BE ACTIVATED
C       IF THE C IN COLUMN 1 IS REMOVED FROM THE TWO CARDS.
C       PRINT 14
C       CALL FRACTN((M+MX),B,(N+NX),A)
        NP=M+MX
        NQ=N+NX
        CALL GENALG(N,NP,NQ,B,A,JM,C)
      5 PRINT16
        DO 6 I=1,N
```

```
      II=I-1
      YP(I)=YL(I)
      J=JM+II-N+1
      IF(J.GE.0) YP(I)=YP(I)+C(J+1)
    6 PRINT 17,II,YL(I),YP(I)
      PRINT 18
      DO 9 I=1,N
      II=I-1
      IF((JM+II-N).GE.0) GO TO 7
      PRINT 19,II
      GO TO 9
    7 PRINT 20,II
      J=JM+I-N
      DO 8 JJ=1,J
      NORD=J-JJ
    8 PRINT 21,NORD,C(JJ)
    9 CONTINUE
      GO TO 1
   10 FORMAT(1H1,17HTRANSFER FUNCTION/)
   11 FORMAT(1H ,9H  T(S) = )
   12 FORMAT(/1H0,5HINPUT/)
   13 FORMAT(1H ,9H  X(S) = )
   14 FORMAT(1H0,9H  F(S) = )
   16 FORMAT(1H0,32HINITIAL CONDITIONS OF THE OUTPUT//13H   DERIVATIVE,8
     1X,9HLEFT HAND,6X,10HRIGHT HAND)
   17 FORMAT(1H ,8X,I2,9X,F10.2,5X,F10.2)
   18 FORMAT(1H0,22HIMPULSES AT THE ORIGIN/)
   19 FORMAT(1H0,3H  (,I2,1H),/8H    Y (T),5X,11HNO IMPULSES)
   20 FORMAT(1H0,3H  (,I2,1H),/,8H    Y (T)10X,22HOF ORDER AND MAGNITUDE)
   21 FORMAT(1H ,20X,I2,7X,F10.2)
   30 FORMAT(2I2)
   31 FORMAT(12F6.2)
   32 FORMAT(I2)
   33 FORMAT(3F6.2,I2)
      END
```

```
      SUBROUTINE ALPH(A,N,F,IDEX)
C
C     THIS SUBROUTINE GENERATES THE ALPHANUMERIC FIELD CORRESPONDING TO
C     A POLYNOMIAL IN  S
C
C        A(N+1)S**N  +  A(N)S**N-1  + ... +  A(1)
C
C     IN THE ARRAY F(I) STARTING AT THE ELEMENT HAVING THE SUBSRIPT,
C     INDEX, AND CONTINUING IN ELEMENTS OF INCREASING VALUES OF
C     SUBSCRIPT.  THE RETURNED VALUE OF INDEX IS THE VALUE OF THE
C     SUBSCRIPT CORRESPONDING TO THE NEXT ELEMENT OF F(I) AFTER THE LAST
C     ONE FILLED BY THE ALPHANUMERIC FIELD GENERATED.
C        THE DECIMAL PART OF THE COEFFICIENTS ARE REPRESENTED TO TWO
C     SIGNIFICANT DIGITS AND THE REMAINDER IS SUPRESSED.  ANY COEFFICIENT
C     OF  VALUE LESS THAN  .01  CAUSES THE CORRESPONDING TERM TO BE
C     SUPPRESSED IN THE ALPHANUMERIC REPRESENTATION. SUBROUTINE  NUM
C     ACCOMPLISHES THE ALPHANUMERIC GENERATION FOR THE COEFFICIENT PART
C     EACH TERM OF THE POLYNOMIAL  A(I).
C
      DIMENSION A(100),AL(20),F(650)
      DATA AL(1)/'0'/,AL(2)/'1'/,AL(3)/'2'/,AL(4)/'3'/,AL(5)/'4'/,AL(6)/
     1'5'/,AL(7)/'6'/,AL(8)/'7'/,AL(9)/'8'/,AL(10)/'9'/,AL(11)/' '/,AL(1
     12)/'S'/,AL(13)/'*'/,AL(14)/'+'/,AL(15)/'-'/,AL(16)/'.'/
      DO 1 I=1,600
    1 F(I)=AL(11)
      NDEX=N+1
      IDEX=2
      IF(N.EQ.0) GO TO 9
    2 IF(A(NDEX).EQ.0.0) GO TO 4
      VAL=A(NDEX)
      IF(VAL.LT.0.0) F(IDEX)=AL(15)
      IDEX=IDEX+2
      VAL=ABS(VAL)
      IF(VAL.EQ.1.0) GO TO 3
      CALL NUM(VAL,IDEX,F,AL)
      IF(VAL.EQ.0.0) IDEX=IDEX-2
      IF(VAL.EQ.0.0) GO TO 4
    3 F(IDEX)=AL(12)
      J=(NDEX-1)/10
      F(IDEX+1)=AL(J+1)
      J=(NDEX-1)-J*10
      F(IDEX+2)=AL(J+1)
      F(IDEX+3)=AL(11)
      F(IDEX+4)=AL(14)
      IDEX=IDEX+4
    4 NDEX=NDEX-1
      IF(NDEX.NE.1) GO TO 2
    9 VAL =A(1)
      IF(VAL)5,7,6
    5 F(IDEX)=AL(15)
    6 IDEX=IDEX+2
      VAL=ABS(VAL)
      CALL NUM(VAL,IDEX,F,AL)
```

```
      IF(VAL.NE.0.0) GO TO 8
      IDEX=IDEX-2
    7 F(IDEX)=AL(11)
    8 IDEX=IDEX+1
      RETURN
      END


      SUBROUTINE NUM(VAL,IDEX,F,AL)
C
C        SUBROUTINE NUM GENERATES AN ALPHANUMERIC FIELD REPRESENTING
C     THE VALUE OF THE ARGUEMENT, VAL. THIS FIELD IS PLACED IN THE
C     ARRAY. F(I), STARTING AT THE ELEMENT, F(INDEX), AND CONTINUING IN
C     INCREASING ELEMENT SUBSCRIPT VALUES. NO GENERATION RESULTS FOR
C     TERMS HAVING COEFFICIENT VALUES OF LESS THAN  .01. THE VALUE OF
C     ARGUEMENT INDEX RETURNED IS OF VALUE ONE GREATER THAN THE LAST
C     FILLED SUBSCRIPT. THE ARRAY ARGUEMENT, A(I), CONTAINS THE LITERAL
C     CONSTANTS OF THE SYMBOLS USED BY THIS SUBROUTINE..... SEE
C     SUBROUTINE ALPH FOR THE NECESSARY LITERAL CONSTANTS THAT MUST
C     BE TRANSFERRED TO SUBROUTINE NUM.
C
      DIMENSION F(610),AL(20)
      R=VAL
      K=ALOG10(VAL)+.0001
      IF(K.LT.(-2)) GO TO 1
      IF(K.LT.0) GO TO 2
      K=K+1
      DO 3 I=1,K
      ZI=K-I
      J=VAL/(10.0**ZI)
      AJ=J
      F(IDEX)=AL(J+1)
      IDEX=IDEX+1
    3 VAL=VAL-AJ*(10.0**ZI)
    2 F(IDEX)=AL(16)
      IDEX=IDEX+1
      J=VAL*10.0
      AJ=J
      F(IDEX)=AL(J+1)
      IDEX=IDEX+1
      J=(VAL*10.0-AJ)*10.0
      F(IDEX)=AL(J+1)
      VAL=R
      IDEX=IDEX+1
      RETURN
    1 VAL=0.0
      RETURN
      END
```

```
      SUBROUTINE XPRINT(INP,A,B,G,L)
C
C        THIS SUBROUTINE CAUSES THE INPUT DESIGNATED THROUGH THE
C     ARGUMENT INP TO BE PRINTED IN CONVENTIONAL FORM. THE ARGUEMENTS
C     A,B,G, AND L CORRESPOND TO THE CONSTANTS ALPHA, BETA, GAMMA, AND L
C     OF TABLE 1 IN THIS THESIS. THE INPUT NUMBER CORRESPONDS TO THE
C     INPUT NUMBERS OF TABLE 1.
C
      GO TO(1,2,3,4,5,6,7,8,9,10,11,12),INP
    1 PRINT 21,L,A
      RETURN
    2 PRINT 22,L,A
      RETURN
    3 PRINT 23,B,A
      RETURN
    4 PRINT 24,A,B
      RETURN
    5 PRINT 25,A,B
      RETURN
    6 PRINT 26,B,A,G
      RETURN
    7 PRINT 27,A,B
      RETURN
    8 PRINT 28,A,B
      RETURN
    9 PRINT 29,A,B
      RETURN
   10 PRINT 30,B,A
      RETURN
   11 PRINT 31,B,A,G
      RETURN
   12 PRINT 32,A,B
      RETURN
   21 FORMAT(1H0,13X,1H(,I2,1H)/6H X(T)=F6.2,7HDEL  (T)/)
   22 FORMAT(1H0,12X,I2/6H  X(T)=F6.2,1HT/)
   23 FORMAT(1H0,13X,F6.2,1HT/6H  X(T)=F6.2,3H XP/)
   24 FORMAT(1H0,5HX(T)=F6.2,4HCOS(F6.2,2HT)/)
   25 FORMAT(1H0,5HX(T)=F6.2,5HCOSH(,F6.2,2HT)/)
   26 FORMAT(1H0,13X,F6.2,1HT/,6H  X(T)=F6.2,9HEXP   COS(,F6.2,2HT)/)
   27 FORMAT(1H0,5HX(T)=F6.2,7H T SIN(,F6.2,2HT)/)
   28 FORMAT(1H0,5HX(T)=F6.2,4HSIN(,F6.2,2HT)/)
   29 FORMAT(1H0,5HX(T)=F6.2,5HSINH(,F6.2,2HT)/)
   30 FORMAT(1H0,16X,F6.2,1HT,/6H  X(T)=F6.2,6H T EXP/)
   31 FORMAT(1H0,13X,F6.2,1HT,/6H  X(T)=F6.2,9HEXP   SIN(F6.2,2HT)/)
   32 FORMAT(1H0,5HX(T)=F6.2,7H T COS(,F6.2,2HT)/)
      END
```

```
      SUBROUTINE DIRALC(ALPHA,L,BETA,G,INP,M,N,B,A,JM,C)
C
C        THIS SUBROUTINE REPRESENTS THE DIRECT ALGORITHM OF TABLE 1.
C     THE ARGUMENTS ALPHA, BETA, G, AND L ARE THE CONSTANTS OF THE
C     ALGORITHM ASSOCIATED WITH THE INPUT FUNCTION DESIGNATED BY INP.
C     M,N,B, AND A ARE THE SCALAR AND ARRAY CONSTANTS ASSOCIATED WITH
C     THE TRANSFER FUNCTION
C
C                     B(M+1)S**M   +  B(M)S**M-1   + ... +   B(1)
C                     ------------------------------------------
C                     A(N+1)S**N   +  A(N)S**N-1   + ... +   A(1)
C
C        THE RETURNED ARRAY, C(I), REPRESENTS THE CONSTANTS COMPUTED BY
C     THE SUBROUTINE USING THE APPROPRIATE ALGORITHM. THE MODIFIED
C     INITIAL CONDITIONS AS WELL AS THE IMPULSE INFORMATION CAN BE
C     FOUND DIRECTLY FROM THE CONSTANTS, C(I). A FLOWCHART OF THIS
C     SUBROUTINE APPEARS IN THIS APPENDIX.
C
      DIMENSION A(100),B(100),C(100)
      A1=ALPHA
      JM=M-1
      N=0
C     DETERMINE JM AND A1
      IF(INP.EQ.1) JM=N+L
      IF(INP.EQ.2) JM=N-L-1
      IF(INP.EQ.7) JM=N-3
      IF(INP.GT.7) JM=N-2
      IF(INP.EQ.2) A1=A1*AFACT(L)
      IF(INP.EQ.7) A1=A1*BETA*2.0
      IF(INP.EQ.11) A1=A1*G
      IF((INP.EQ.8).OR.(INP.EQ.9)) A1=A1*BETA
      C(1)=A1*B(M+1)
      IF(JM.LT.1) RETURN
      A(N+1)=1
      GO TO(21,21,22,23,23,26,29,23,23,25,26,28),INP
C     NOW C(J) IS TO BE COMPUTED
   21 DO 5 J=1,JM
      C(J+1)=0
      IF(M.GE.J) C(J+1)=A1*B(M+1-J)
      DO 6 I=1,J
      IF(N.LT.I) GO TO 5
    6 C(J+1)=C(J+1)-C(J+1-I)*A(N+1-I)
    5 CONTINUE
      RETURN
   22 DO 7 J=1,JM
      C(J+1)=A1*B(M+1-J)
      DO 8 I=1,J
      IF(N-I)9,8,8
    9 C(J+1)=C(J+1)+BETA*A(1)
      GO TO 7
    8 C(J+1)=C(J+1)+(BETA*A(N+2-I)-A(N+1-I))*C(J+1-I)
    7 CONTINUE
      RETURN
```

```
   23 BB=BETA*BETA
      IF((INP.EQ.4).OR.(INP.EQ.9)) BB=-BB
      DO 10 J=1,JM
      C(J+1)=A1*B(M+1-J)-C(J)*A(N)
      IF(J.EQ.1) GO TO 10
      DO 11 I=2,J
      IF((N+2).LT.I) GO TO 10
      V=BB*A(N+3-I)
      IF(N.GE.I) V=V+A(N+1-I)
   11 C(J+1)=C(J+1)-C(J+1-I)*V
   10 CONTINUE
      RETURN
   25 G=0
   26 BB=BETA*BETA+G*G
      DO 12 J=1,JM
      C(J+1)=A1*B(M+1-J)-C(J)*(A(N)-2.0*BETA)
      IF(INP.EQ.6)  C(J+1)=C(J+1)-A1*BETA*B(M+2-J)
      IF(J.EQ.1) GO TO 12
      DO 13 I=2,J
      IF((N+2).LT.I) GO TO 12
      V=BB*A(N+3-I)
      IF((N+1).GE.I) V=V-2.0*BETA*A(N+2-I)
      IF(N.GE.I) V=V+A(N+1-I)
   13 C(J+1)=C(J+1)-C(J+1-I)*V
   12 CONTINUE
      RETURN
   28 V=-BETA*BETA
   29 C(2)=A1*B(M)-C(1)*A(1)
      IF(JM.EQ.1) RETURN
      DO 14 J=2,JM
      C(J+1)=A1*(B(M+1-J)+V*B(M+3-J))-C(J)*A(N)
      DO 15 I=2,J
      IF((N+4).LT.I) GO TO 14
      V=0
      IF(I.GT.3) V=V+(BETA**4)*A(N+5-I)
      IF((N+2).GE.I) V=V+2.0*A(N+3-I)*BETA*BETA
      IF(N.GE.I) V=V+A(N+1-I)
   15 C(J+1)=C(J+1)- C(J+1-I)*V
   14 CONTINUE
      RETURN
      END


      FUNCTION AFACT(N)
C
C        THIS FUNCTION COMPUTES N FACTORIAL FOR NONNEGATIVE NUMBERS.
C     IF THE VALUE OF N IS NEGATIVE, THE RETURNED VALUE OF AFACT IS
C     ZERO. THIS IS A REAL-VALUED FUNCTION SUBPROGRAM.
C
      LL=N
      LLL=1
    1 IF(LL) 2,3,4
    2 AFACT=0
      RETURN
    3 AFACT=LLL
      RETURN
    4 LLL=LLL*LL
      LL=LL-1
      GO TO 1
      END
```

```
      SUBROUTINE FRACTN(M,B,N,A)
C
C     THIS SUBROUTINE CAUSES THE FRACTION OF TWO POLYNOMIALS IN  S,
C
C        B(M+1)S**M  +  B(M)**M-1  + ...+  B(1)
C        ----------------------------------------
C        A(M+1)S**M  +  A(N)S**N-1  + ...+  A(1)
C
C     TO BE PRINTED ALPHANUMERICALLY.  IT CALLS SUBROUTINE ALPH TO
C     GENERATE THE ALPHANUMERIC FIELDS FOR EACH POLYNOMIAL SEPERATELY.
C     THE ARGUMENTS  B AND A  IN THE HEADING ARE THE ARRAYS OF EACH
C     POLYNOMIAL.  A ZERO COEFFICIENT CAUSES THE CORRESPONDING TERM TO
C     BE SUPPRESSED.
C
      DIMENSION A(100),B(100),F(650),ALINE(200)
      DATA PR/')'/,PL/'('/,BAR/'-'/,AS/'S'/
      J=0
      KP=0
      CALL ALPH(B,M,F,KN)
   25 IF(KN-112)1,2,3
    1 PRINT 30,(F(I),I=2,111)
   30 FORMAT(1H+,9X,111A1)
      GO TO 20
    2 PRINT 30,(F(I),I=2,112)
      GO TO 20
    3 F(1)=PL
      DO 4 I=85,108
      J=I+2
      IF(F(I).EQ.AS) GO TO 5
    4 CONTINUE
    5 PRINT 30,(F(I),I=1,J)
   11 IF(KN-J-112)6,7,8
    6 F(KN)=PR
      J=J+1
      PRINT 31,(F(I),I=J,KN)
   31 FORMAT(1H ,9X,111A1)
      GO TO 20
    7 F(KN)=PR
      J=J+2
      PRINT 31,(F(I),I=J,KN)
      GO TO 20
    8 DO 9 I=85,108
      JJ=I+2
      IF(F(J+I).EQ.AS) GO TO 10
    9 CONTINUE
   10 PRINT 31,(F(J+I),I=1,JJ)
      J=J+JJ
      GO TO 11
   20 IF(KP.EQ.1) RETURN
      KP=1
      IF(N.EQ.0) RETURN
      CALL ALPH(A,N,F,KN)
      KK=KN
```

```
      IF(KN.GT.KM) KK=KN
      KN=KN
      IF(KK.GT.113) KK=113
      KK=KK-2
      DO 21 I=1,KK
   21 ALINE(I)=BAR
      PRINT 31,(ALINE(I),I=1,KK)
      PRINT 32
   32 FORMAT(1H )
      GO TO 25
      END
```

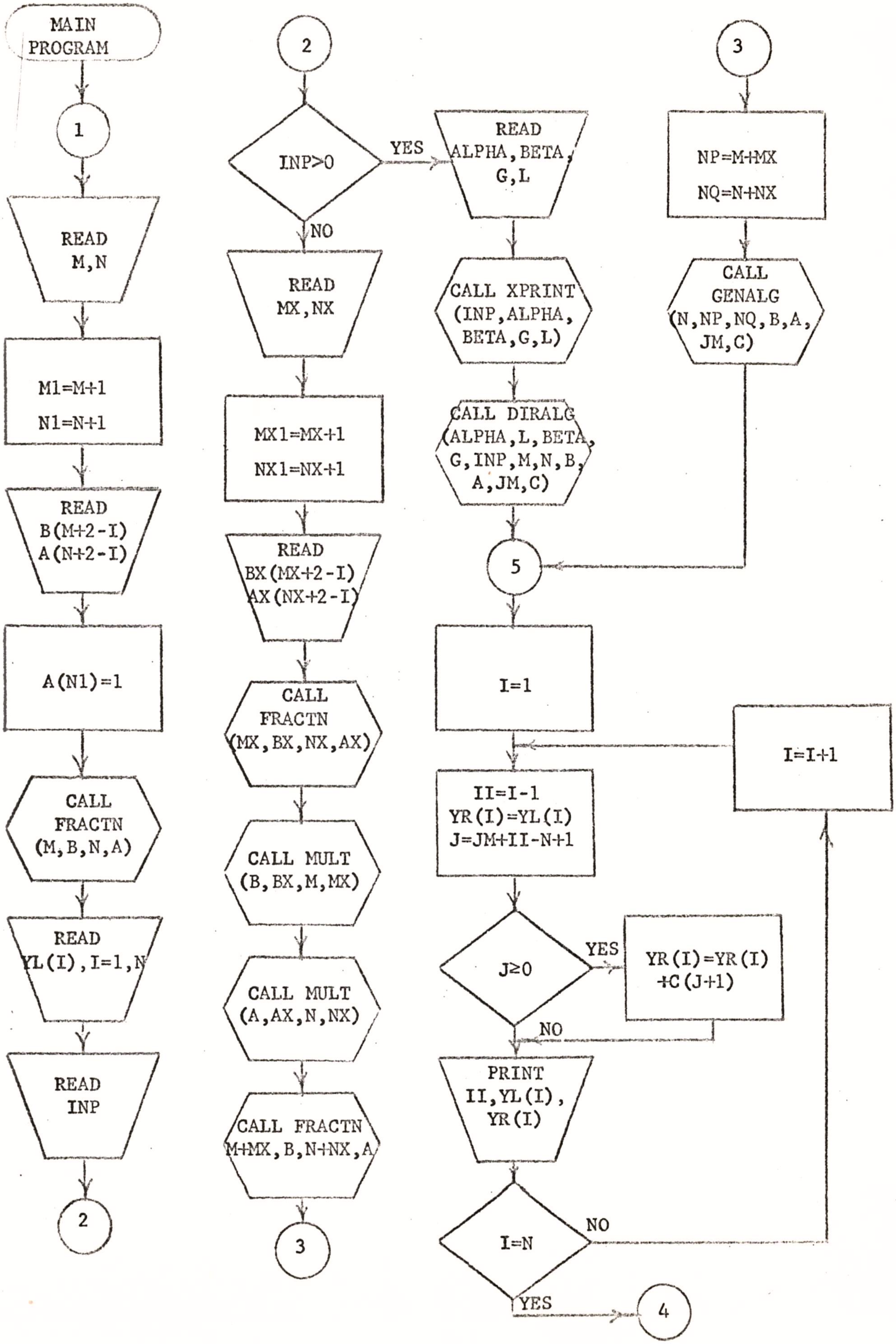
```
      SUBROUTINE MULT(A,B,NA,NB)
C
C        THIS SUBROUTINE CAUSES THE TWO POLYNOMIALS IN S, A(S) AND B(S),
C     TO BE MULTIPLIED TOGETHER. THE RESULTING POLYNOMIAL IS TRANSFERRED
C     TO THE ARRAY A(I) IN TERMS OF ITS COEFFICIENTS. ( THE TERM OF
C     POWER 'J' IS REPRESENTED BY A(J+1) ) THE VALUES OF NA AND NB
C     CORRESPOND TO THE HIGHEST POWERS OF THE POLYNOMIALS A(S) AND B(S),
C     RESPECTFULLY. THE RETURNED VALUE OF NA IS THAT WHICH CORRESPONDS
C     TO THE RESULTANT POLYNOMIAL.
C
      DIMENSION A(100),B(100),R(100)
      NA1=NA+1
      NB1=NB+1
      NR=NA1+NB1-1
      DO 3 I=1,NR
    3 R(I)=0.0
      DO 1 I=1,NA1
      DO 1 J=1,NB1
    1 R(I+J-1)=A(I)*B(J) +R(I+J-1)
      DO 2 I=1,NR
    2 A(I)=R(I)
      RETURN
      END
```

```
      SUBROUTINE GENALG(N,P,Q,B,A,JM,C)
C
C        SUBROUTINE GENALG IS GIVEN A FUNCTION IN S
C
C                B(P+1)S**P  +  B(P)S**P-1  + ... +  B(1)
C                ----------------------------------------
C                A(Q+1)S**Q  +  A(Q)S**Q-1  + ... +  A(1)
C
C        THIS FUNCTION REPRESENTS THE PRODUCT OF A TRANSFER FUNCTION
C     OF ORDER N AND THE LAPLACE TRANSFORM OF AN INPUT FUNCTION. THE
C     PURPOSE OF THIS SUBROUTINE IS TO EXPAND THE FUNCTION BY SYNTHETIC
C     DIVISION TO
C
C     C(0)S**P-Q  +  C(1)S**P-Q-1  + ... +C(P+N-Q)  +  U(S)/V(S)
C
C     WHERE U(S) AND V(S) ARE POLYNOMIALS IN S. ONLY THE CONSTANT
C     C(I)  ARE COMPUTED BY USE OF AN ALGORITHM. THE ARGUEMENT JM
C     EQUALS P+N-Q. A FLOWCHART OF THIS SUBROUTINE APPEARS IN THIS
C     APPENDIX.
C
      DIMENSION A(100),B(100),C(100)
      INTEGER P,Q
      JM=P+N-Q
      C(1)=B(P+1)
      IF(JM.LE.0) RETURN
      DO 1 J=1,JM
      C(J+1)=B(P+1-J)
      DO 2 I=1,J
      IF(I.GT.Q) GO TO 1
    2 C(J+1)=C(J+1)-C(J+1-I)*A(Q+1-I)
    1 CONTINUE
      RETURN
      END
```

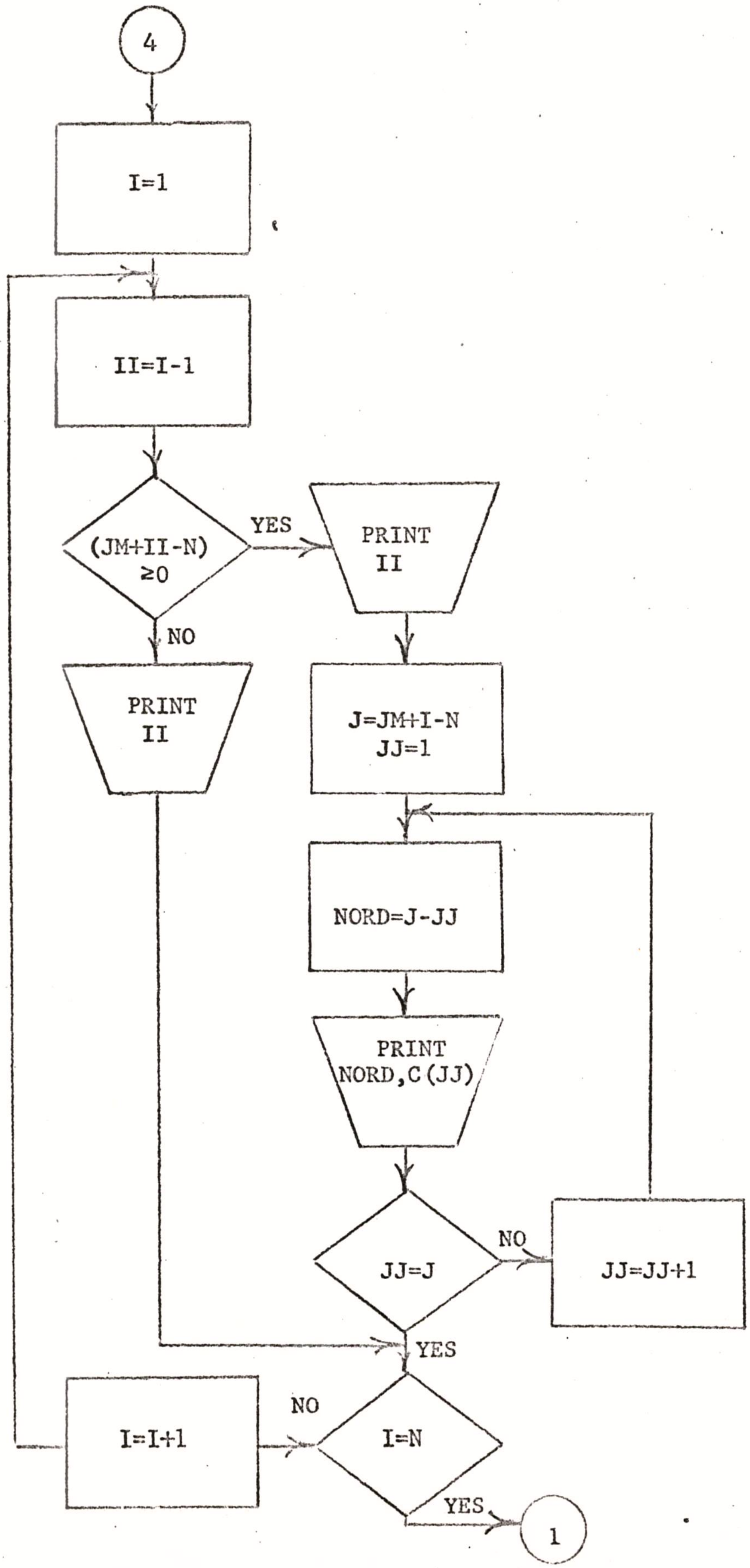
A3.4 Flowcharts

Flowcharts of the main program, subroutine DIRALG, and subroutine GENALG follow. No flowcharts for the other subroutines and the function subprograms are included since these programs are concerned only with formating of the output and performing operations of secondary importance.

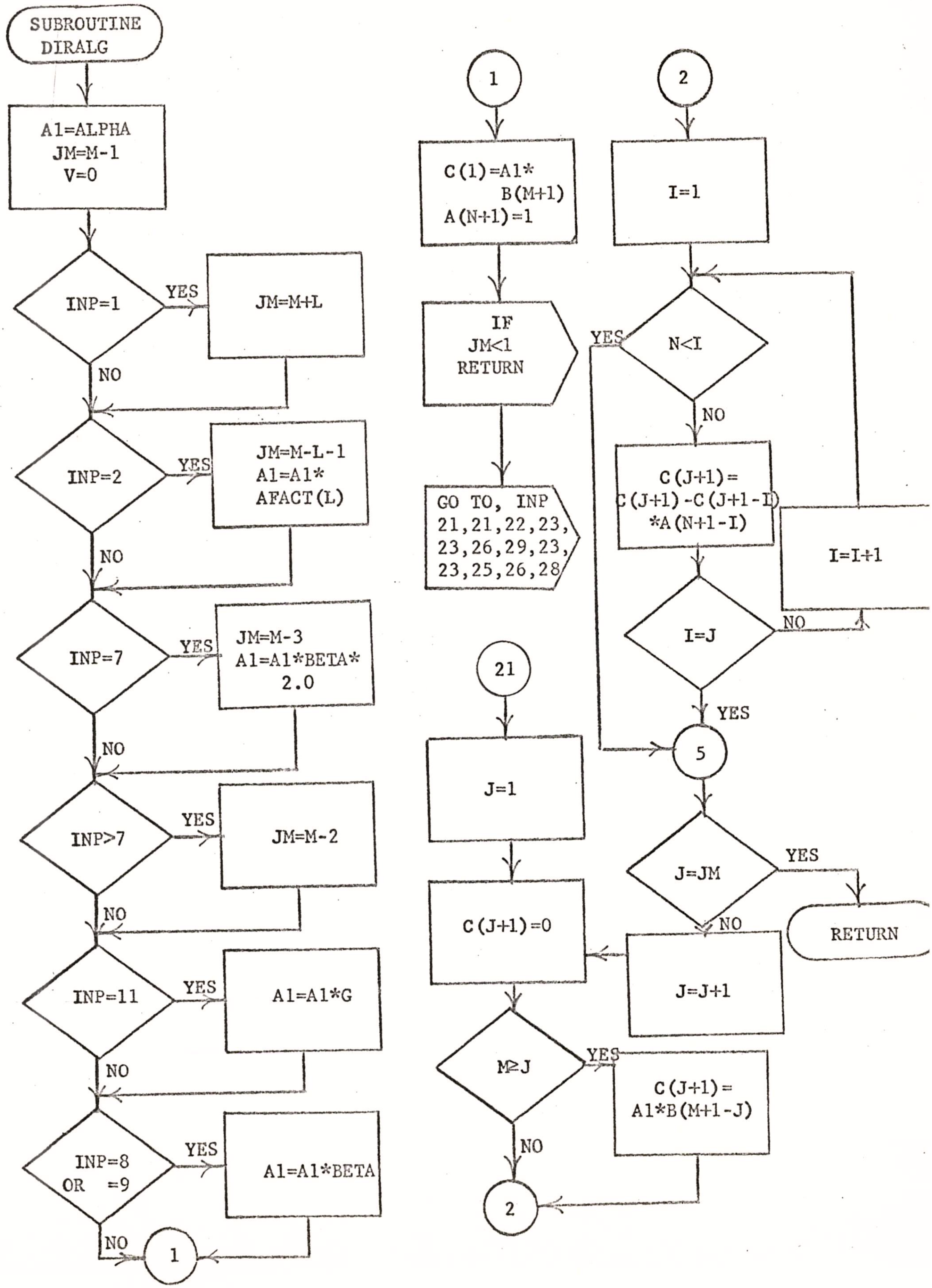
A3.4.1 Flowchart of Main Program



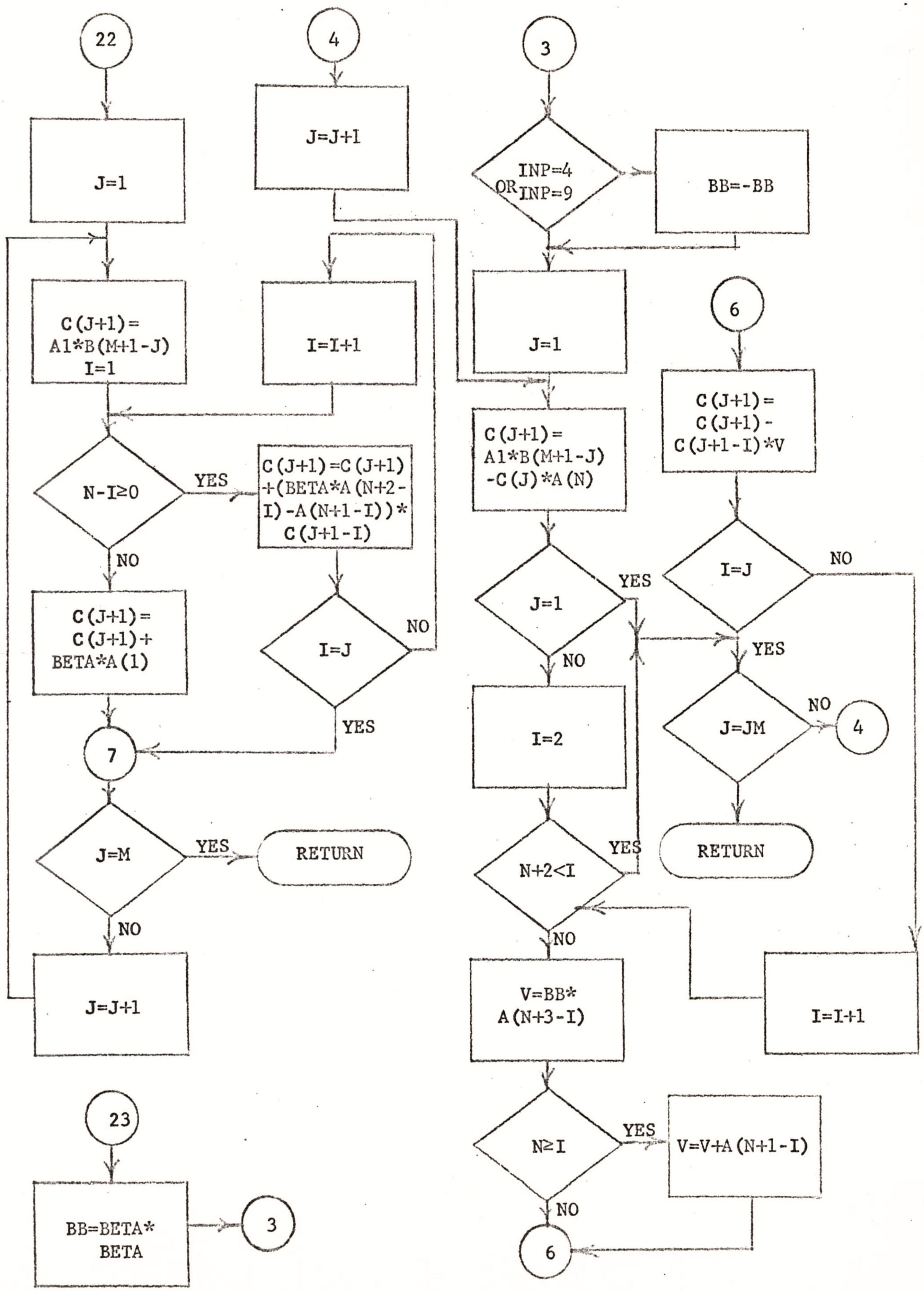
A3.4.1 Flowchart of Main Program (continued)



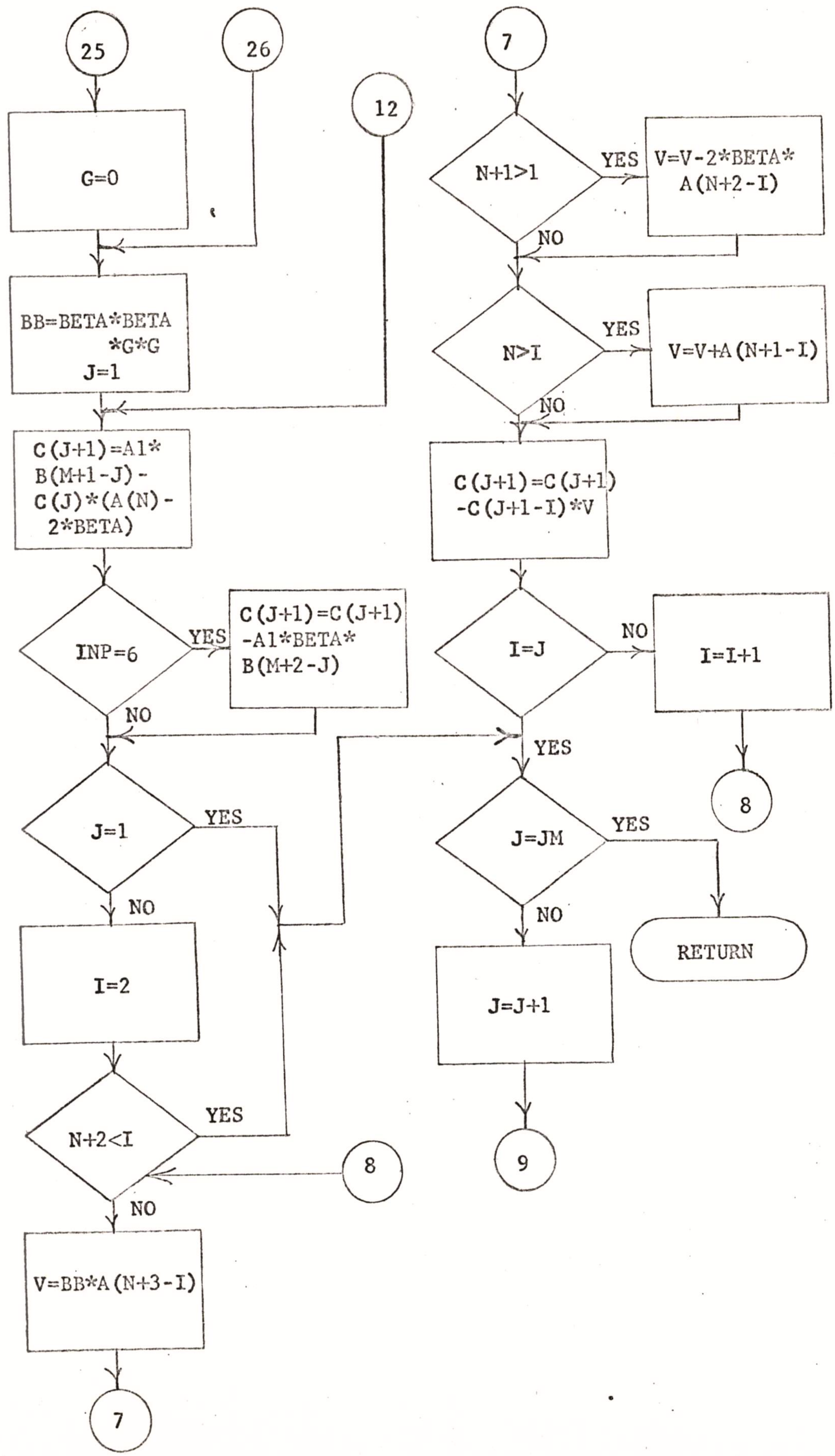
A3.4.2 Flowchart of Subroutine DIRALG



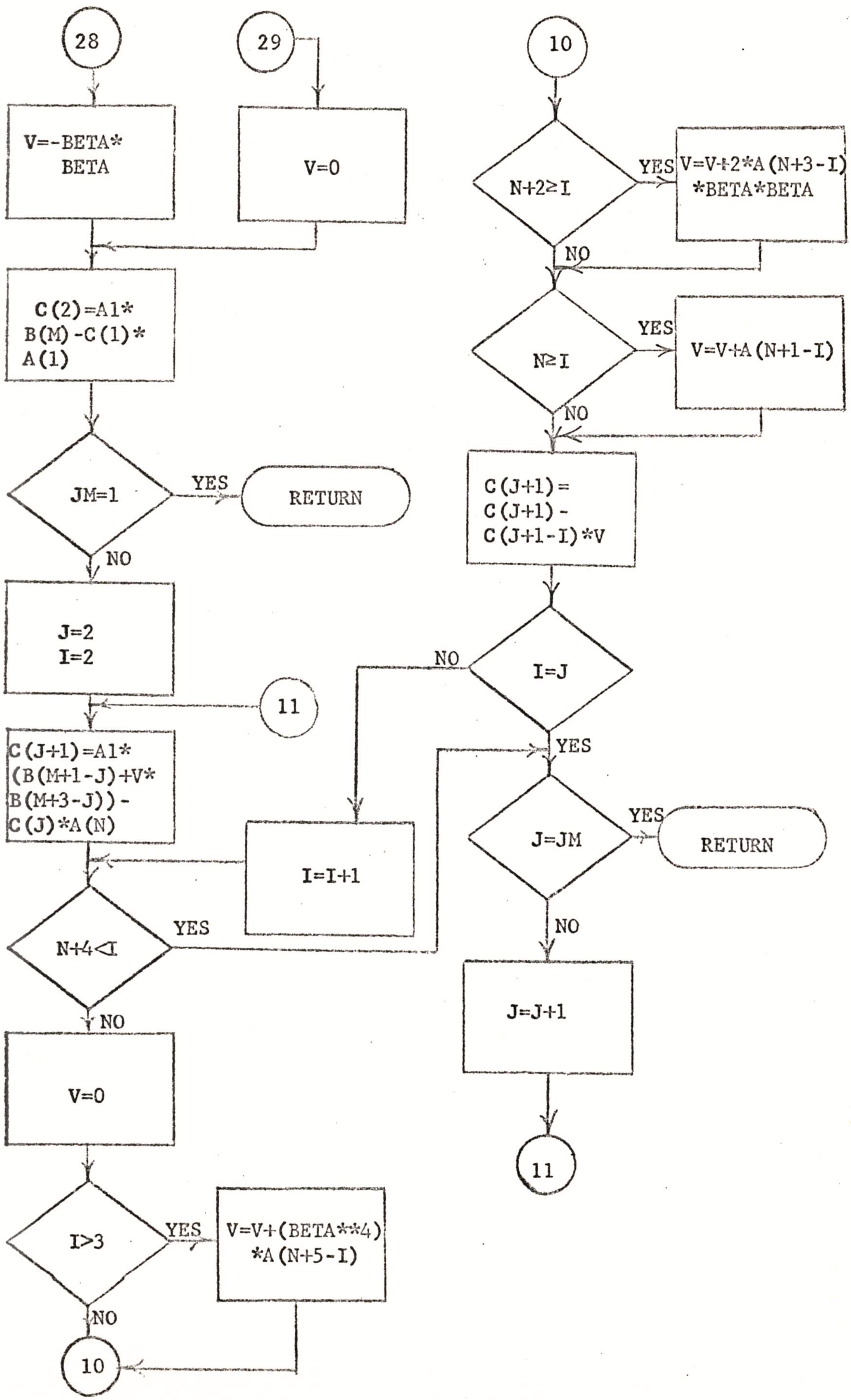
A3.4.2 Flowchart of Subroutine DIRALG (continued)



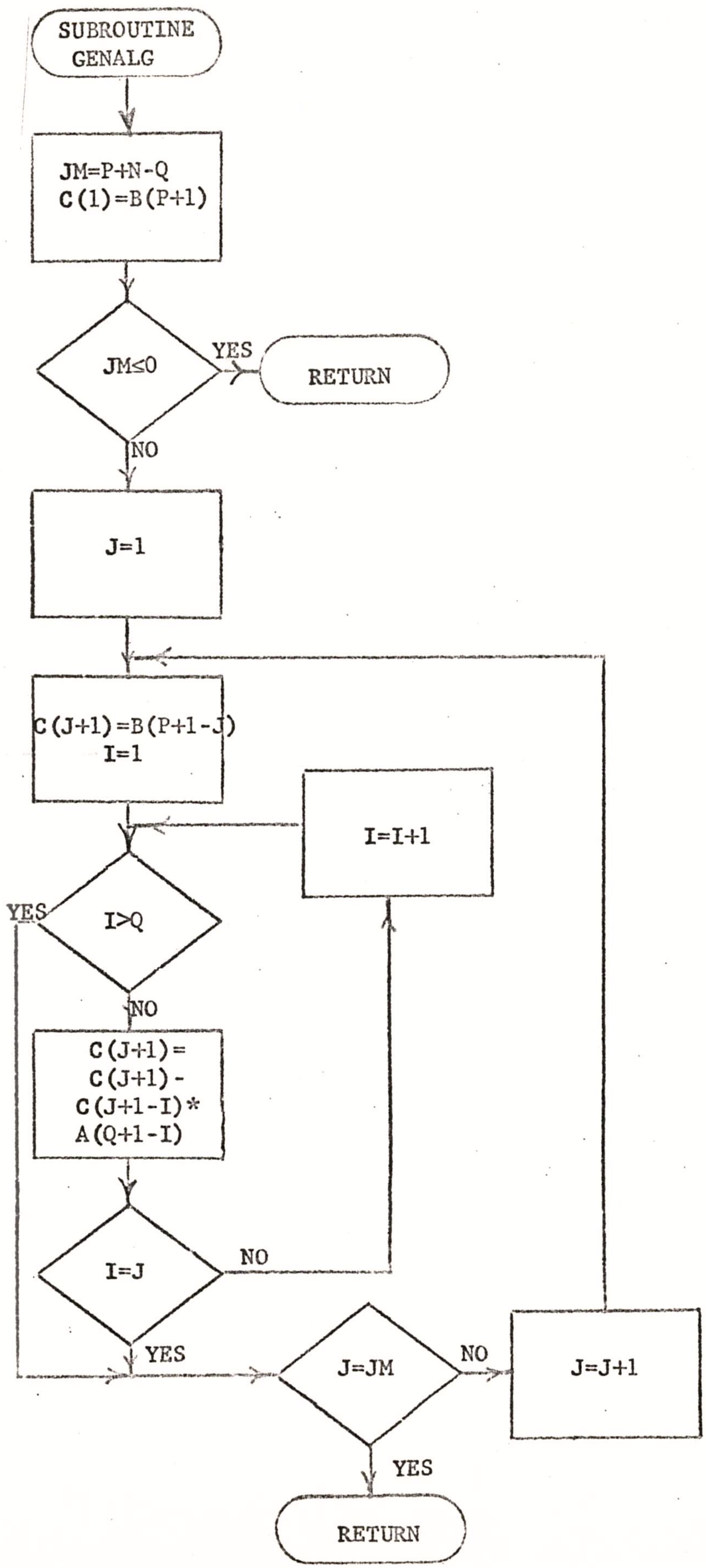
A3.4.2 Flowchart of Subroutine DIRALG (continued)



A3.4.2 Flowchart of Subroutine DIRALG (continued)



A3.4.3 Flowchart of Subroutine GENALG



A3.5 Sample Runs

This section contains seven sample runs using the programs of Section 3.5.3 . Runs 1,3,5, and 6 use the general algorithm for modifying the initial conditions. Runs 2,4, and 7 use direct algorithms to solve the same problems as those given in runs 1,3, and 6 respectively.

The data used for each run is shown on the following pages. The results for each run follows these pages.

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Program A3.5.1 Sample Data

Coded By

Checked By

Identification 73 80

Date

Page 1 of 3

C FOR COMMENT

STATEMENT NUMBER | Cont. | FORTRAN STATEMENT

```
01                                                          RUN  1
  1.0   1.0
  1.0
0
0 0
  1.0   1.0
01                                                          RUN  2
  1.0   1.0
  1.0
1
  1.0   0.0   0.0   0
11                                                          RUN  3
  1.0   1.0   2.0
  1.0
0
3 0
  1.0   0.0   0.0   0.0   1.0
```

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Program: Sample Data
Coded By:
Checked By:

Identification (73–80)

Date:
Page 2 of 3

```
C FOR COMMENT
STATEMENT NUMBER | Cont. | FORTRAN STATEMENT
11                                                          RUN  4
  1.0   1.0  2.0
  1.0
1
  1.0   0.0  0.0  3
34                                                          RUN  5
  2.0   3.0  5.0  1.0  0.0  1.0  7.0  -3.0
  1.0   0.0  0.0  -1.0
0
55
  5.0   1.0  1.0  -2.0  1.0  2.0  1.0  1.0  -1.0  -2.0  1.0  1.0


34                                                          RUN  6
  2.0   3.0  5.0  1.0  0.0  1.0  7.0  -3.0
  1.0   0.0  0.0  -1.0
6
  1.0   5.0  2.0  0
```

IBM

FORTRAN CODING FORM

Form X28-7327-3
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Program Sample Data
Coded By
Checked By

Identification 73 80

Date
Page 3 of 3

C FOR COMMENT

STATEMENT NUMBER | Cont. | FORTRAN STATEMENT

```
34                                                          RUN 7
  2.0   3.0  5.0  1.0  0.0  1.0  7.0 -3.0
  1.0   0.0  0.0 -1.0
0
12
  1.0  -5.0  1.0-10.0 29.0
```

A3.5.2 Sample Results

```
TRANSFER FUNCTION

  T(S) =     1.00                              RUN 1
          -------------
            S01 + 1.00


INPUT

  X(S) =     1.00

INITIAL CONDITIONS OF THE OUTPUT

  DERIVATIVE           LEFT HAND        RIGHT HAND
       0                    1.00             2.00

IMPULSES AT THE ORIGIN


  ( 0)
  Y (T)        NO IMPULSES
```

```
TRANSFER FUNCTION

  T(S) =     1.00                              RUN 2
          -------------
            S01 + 1.00


INPUT

               ( 0)
X(T)=   1.00DEL  (T)

INITIAL CONDITIONS OF THE OUTPUT

  DERIVATIVE           LEFT HAND        RIGHT HAND
       0                    1.00             2.00

IMPULSES AT THE ORIGIN


  ( 0)
  Y (T)        NO IMPULSES
```

TRANSFER FUNCTION

```
T(S) =    S01 + 1.00                    RUN 3
        -------------
          S01 + 2.00
```

INPUT

X(S) = S03

INITIAL CONDITIONS OF THE OUTPUT

DERIVATIVE	LEFT HAND	RIGHT HAND
0	1.00	9.00

IMPULSES AT THE ORIGIN

```
( 0)
Y (T)            OF ORDER AND MAGNITUDE
                       3          1.00
                       2         -1.00
                       1          2.00
                       0         -4.00
```

TRANSFER FUNCTION

```
T(S) =    S01 + 1.00                    RUN 4
        -------------
          S01 + 2.00
```

INPUT

```
             ( 3)
X(T)= 1.00DEL (T)
```

INITIAL CONDITIONS OF THE OUTPUT

DERIVATIVE	LEFT HAND	RIGHT HAND
0	1.00	9.00

IMPULSES AT THE ORIGIN

```
( 0)
Y (T)            OF ORDER AND MAGNITUDE
                       3          1.00
                       2         -1.00
                       1          2.00
                       0         -4.00
```

RUN 5

TRANSFER FUNCTION

```
T(S) =    2.00S03 + 3.00S02 + 5.00S01 + 1.00
          ------------------------------------
          S04 + S02 + 7.00S01 - 3.00
```

INPUT

```
X(S) =    5.00S05 + S04 + S03 - 2.00S02 + SC1 + 2.00
          --------------------------------------------
          S05 + S04 - S03 - 2.00S02 + SC1 + 1.00
```

INITIAL CONDITIONS OF THE OUTPUT

DERIVATIVE	LEFT HAND	RIGHT HAND
0	1.00	11.00
1	0.0	7.00
2	0.0	23.00
3	-1.00	-75.00

IMPULSES AT THE ORIGIN

```
( 0)
Y (T)      NO IMPULSES

( 1)
Y (T)            OF ORDER AND MAGNITUDE
                    0            10.00

( 2)
Y (T)            OF ORDER AND MAGNITUDE
                    1            10.00
                    0             7.00

( 3)
Y (T)            OF ORDER AND MAGNITUDE
                    2            10.00
                    1             7.00
                    0            23.00
```

TRANSFER FUNCTION

RUN 6

```
T(S) =   2.00S03 + 3.00S02 + 5.00S01 + 1.00
         ----------------------------------
         S04 + S02 + 7.00S01 - 3.00
```

INPUT

```
                5.00T
X(T)=  1.00EXP   COS(  2.00T)
```

INITIAL CONDITIONS OF THE OUTPUT

DERIVATIVE	LEFT HAND	RIGHT HAND
0	1.00	1.00
1	0.0	2.00
2	0.0	13.00
3	-1.00	59.00

IMPULSES AT THE ORIGIN

```
( 0)
Y (T)      NO IMPULSES

( 1)
Y (T)      NO IMPULSES

( 2)
Y (T)            OF ORDER AND MAGNITUDE
                    0             2.00

( 3)
Y (T)            OF ORDER AND MAGNITUDE
                    1             2.00
                    0            13.00
```

RUN 7

TRANSFER FUNCTION

```
T(S) =   2.00S03 + 3.00S02 + 5.00S01 + 1.00
         -----------------------------------
          S04 + S02 + 7.00S01 - 3.00
```

INPUT

```
X(S) =   S01 - 5.00
         -------------------------
          S02 - 10.00S01 + 29.00
```

INITIAL CONDITIONS OF THE OUTPUT

DERIVATIVE	LEFT HAND	RIGHT HAND
0	1.00	1.00
1	0.0	2.00
2	0.0	13.00
3	-1.00	59.00

IMPULSES AT THE ORIGIN

```
( 0)
Y (T)       NO IMPULSES

( 1)
Y (T)       NO IMPULSES

( 2)
Y (T)              OF ORDER AND MAGNITUDE
                       0              2.00

( 3)
Y (T)              OF ORDER AND MAGNITUDE
                       1              2.00
                       0             13.00
```

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